Contrôle et jeux sur les graphes

Formation sur les Systèmes à Événements Discrets (SED)

2e édition
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Nantes

Société d'Automatique,
de Génie Industriel & de Productique

Introduction

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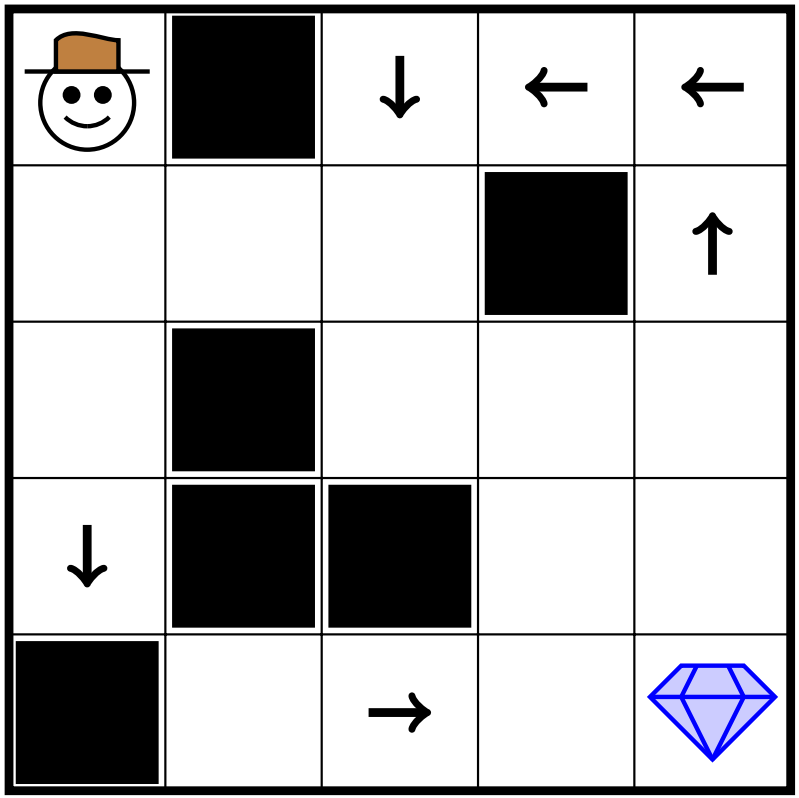
- But our games can be graphs, not only trees !
- Here :
  - One or two-player, win/lose games ;
  - The winning condition is reachability or safety ;
  - Without or with explicit timing constraints.

One-player games and verification

Reachability



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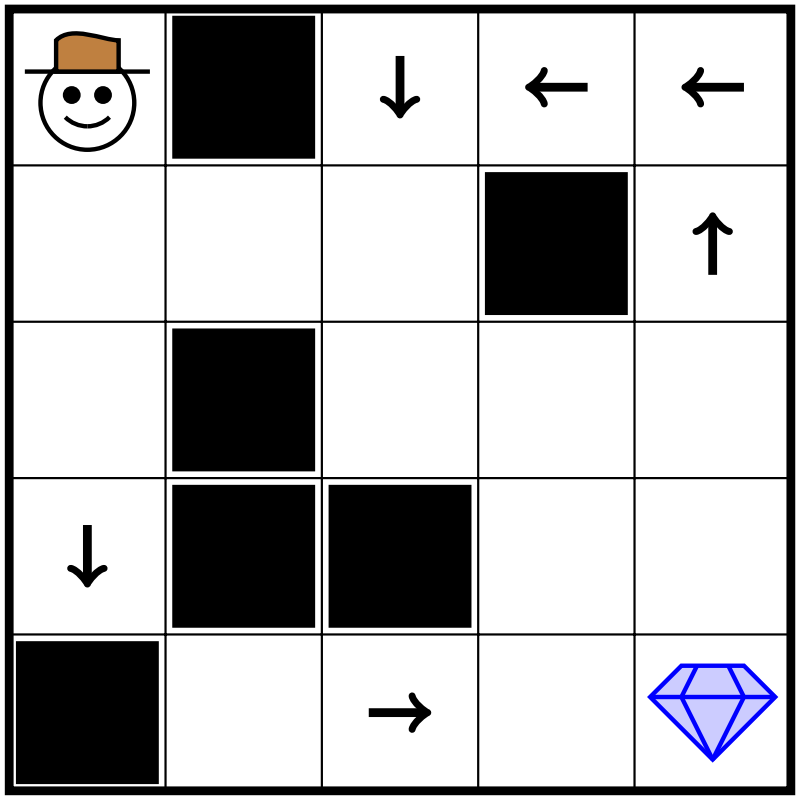
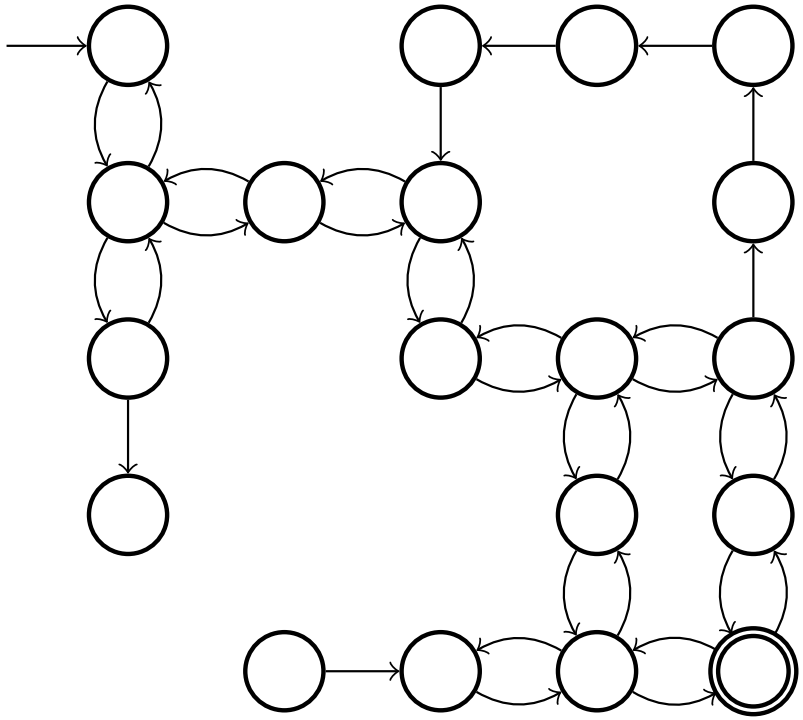
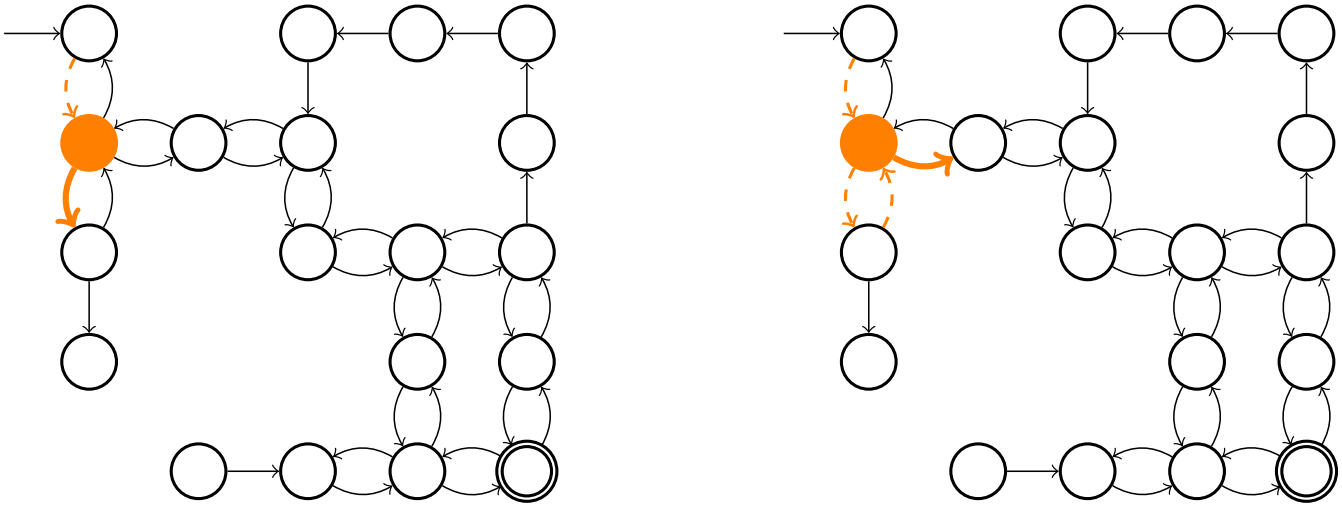




Strategies

- A strategy associates an edge to histories of past states



- A positional/memoryless strategy depends only on the last state in the history : the current state.

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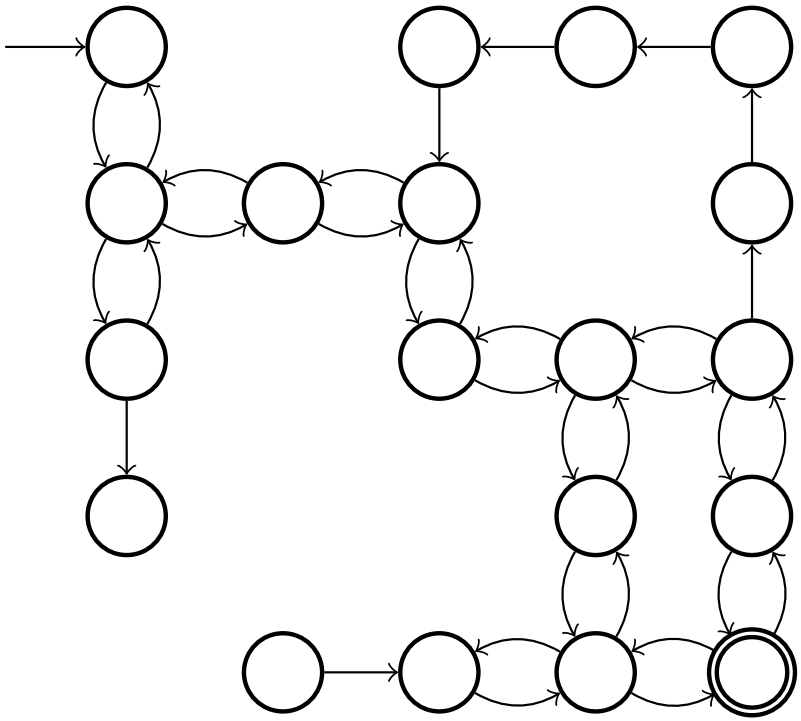
So we only need to find a winning play : a path to the goal !
Let us use Breadth-first Search (symbolic version).

Reachable states



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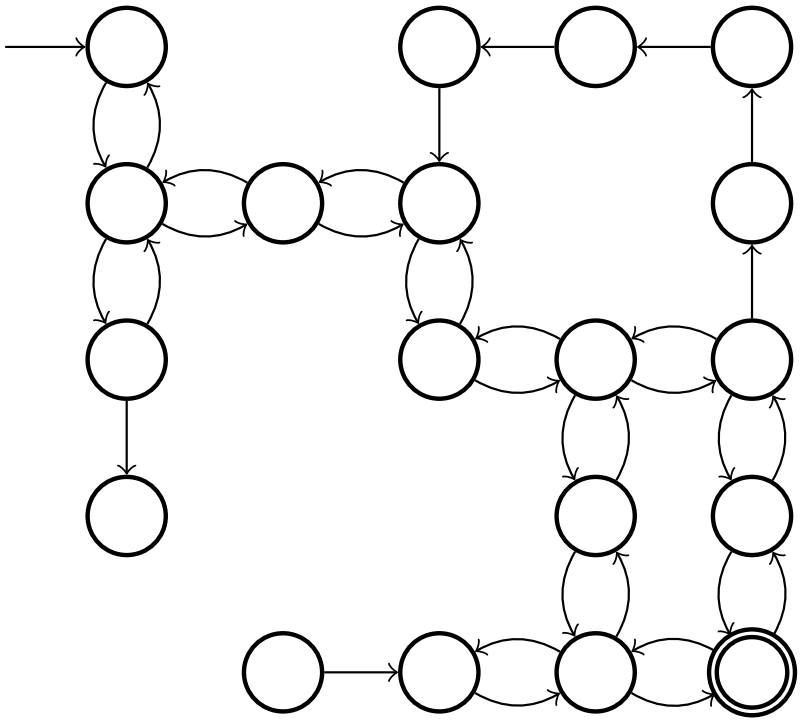

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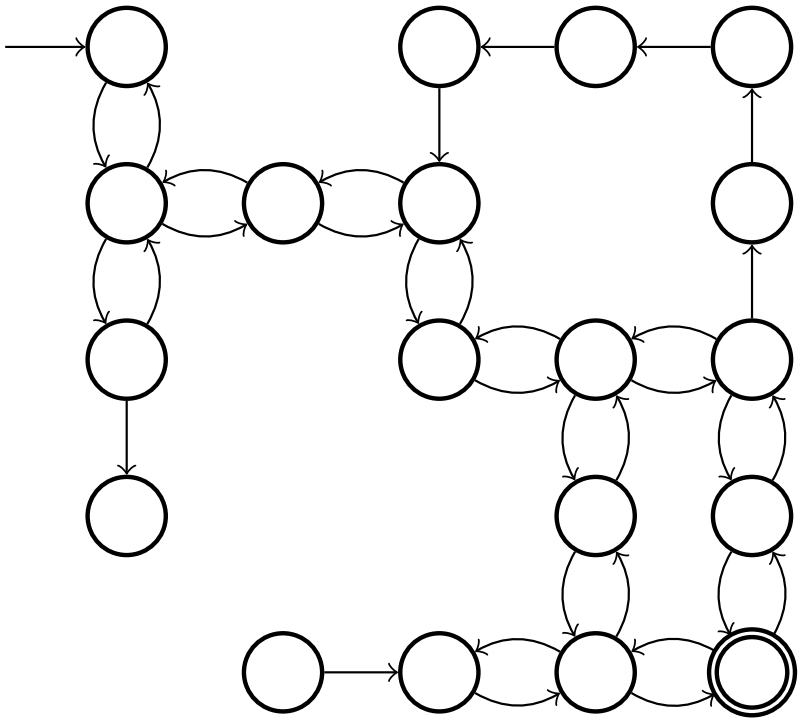

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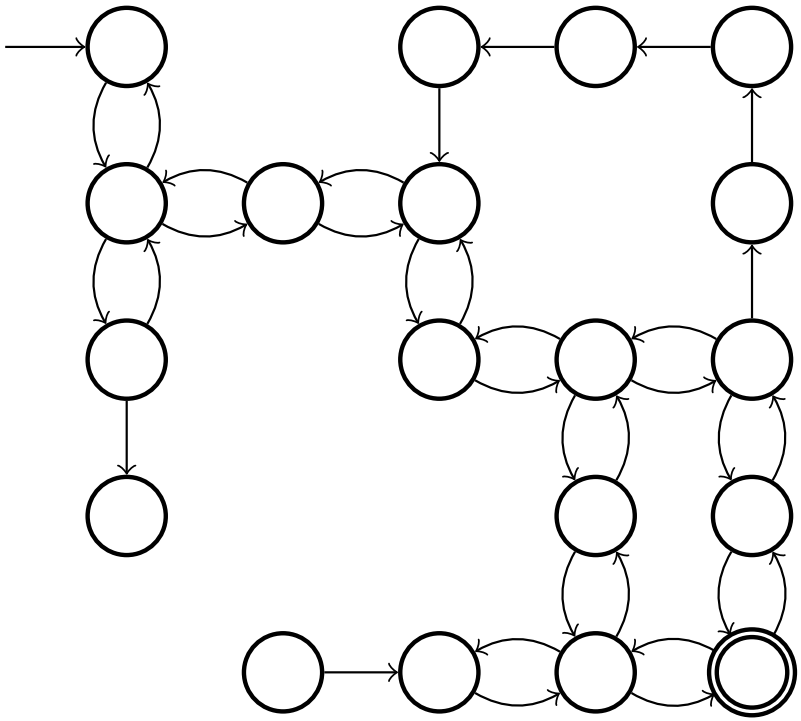

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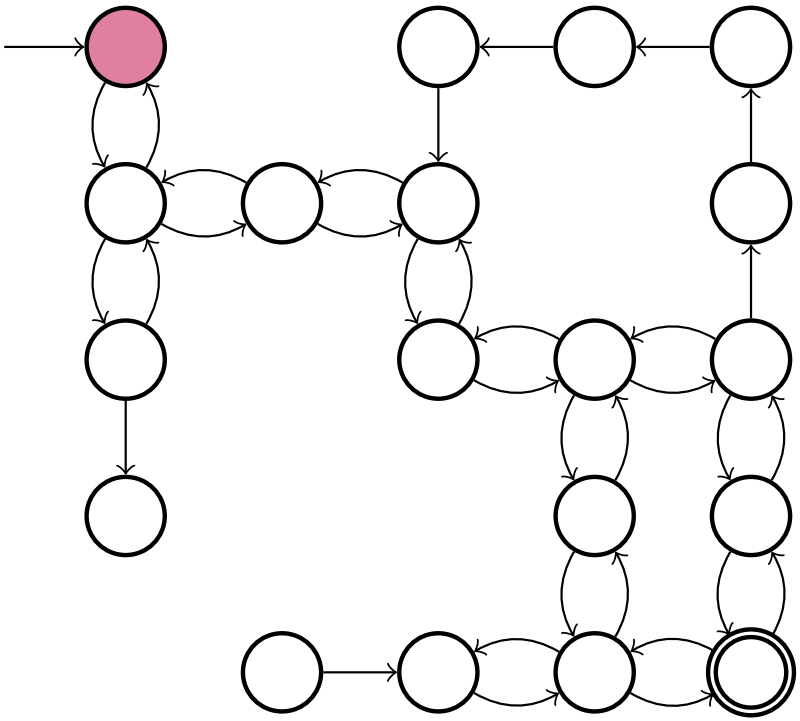

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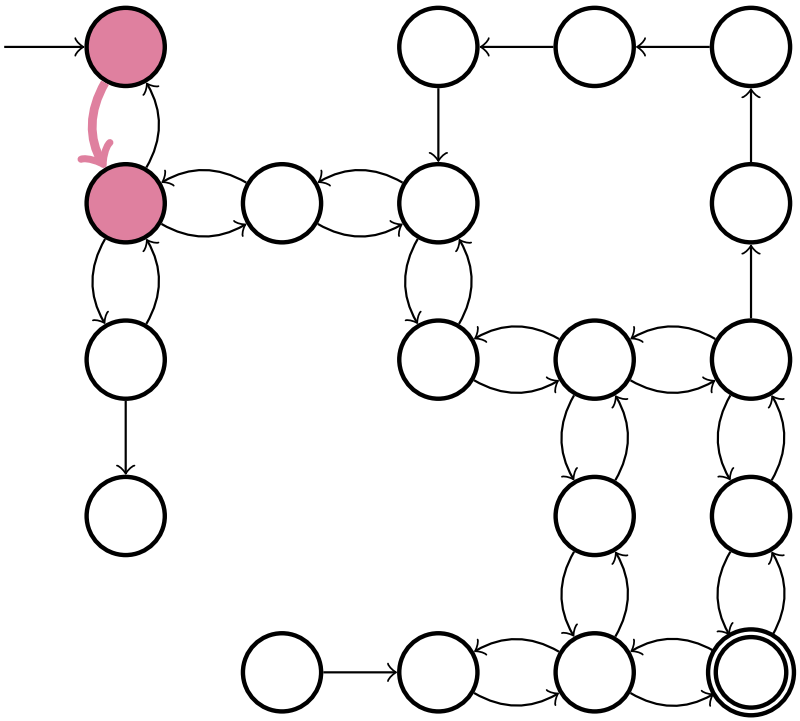

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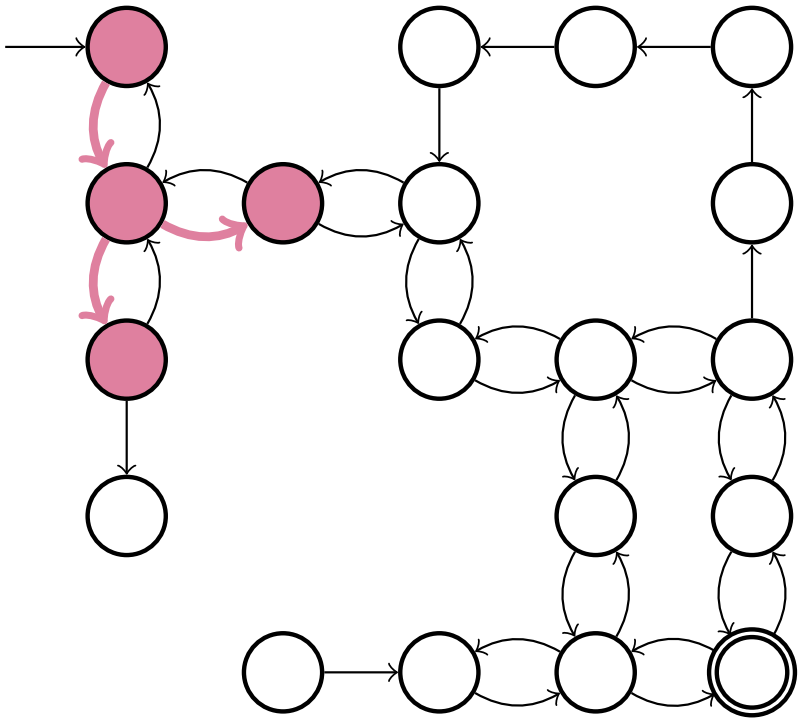

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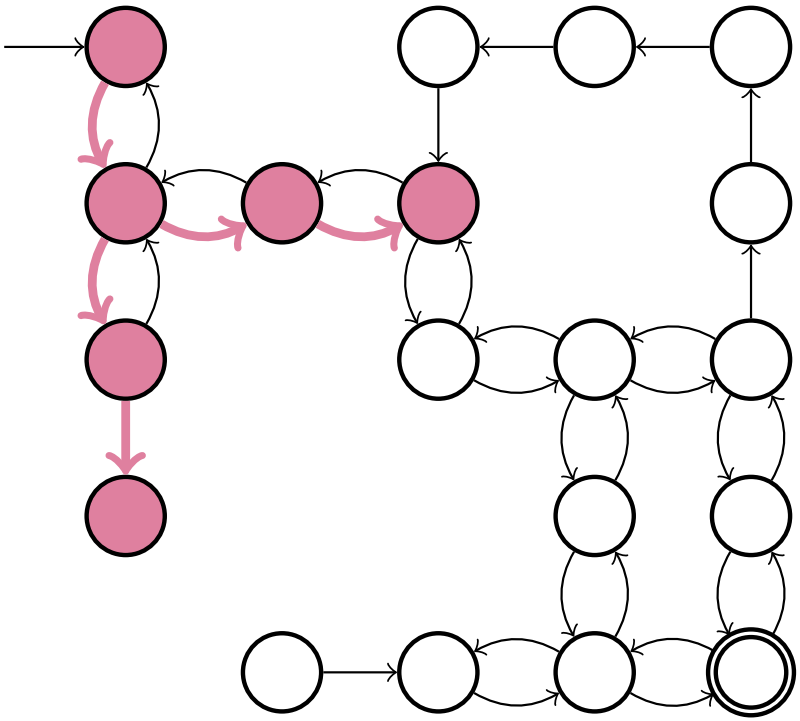

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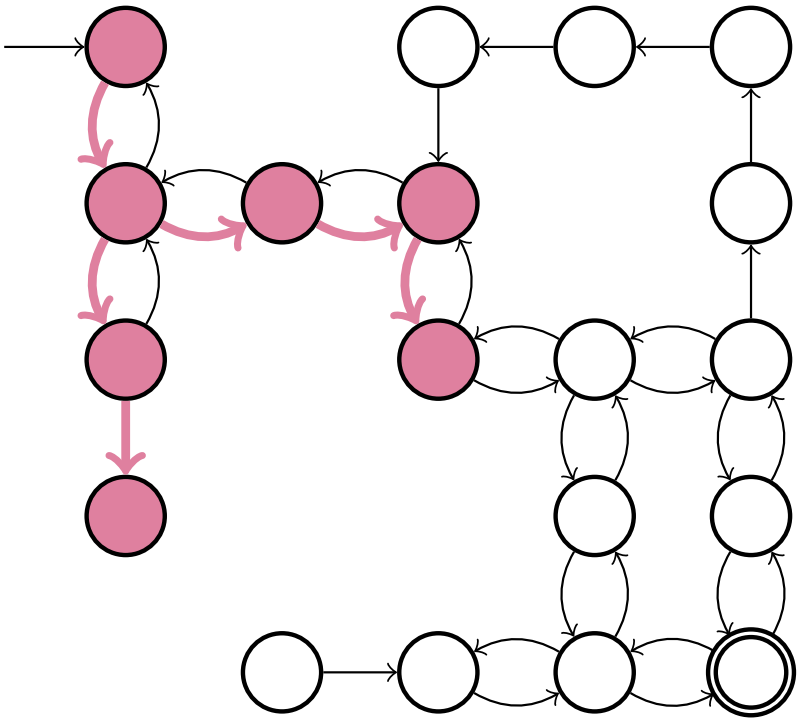

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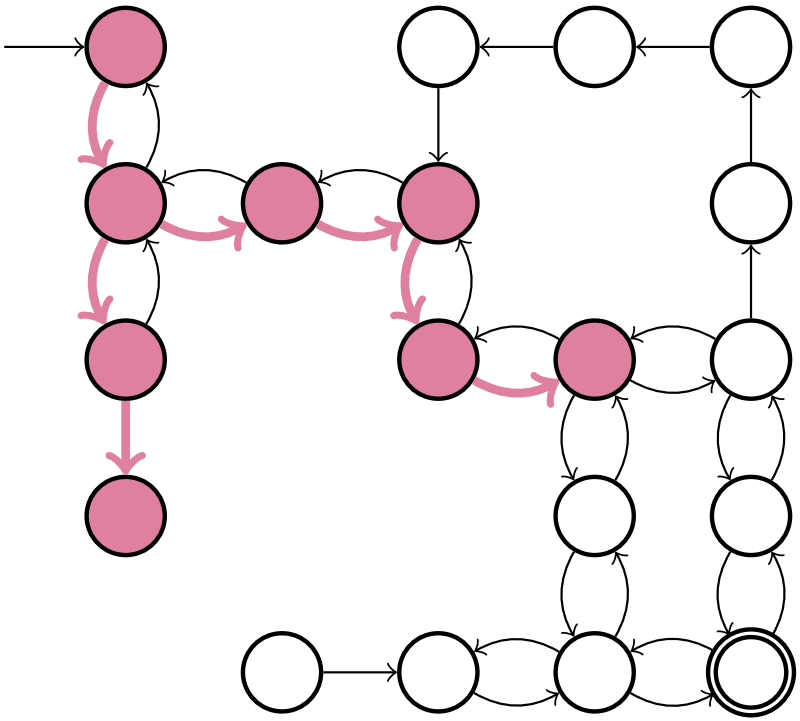

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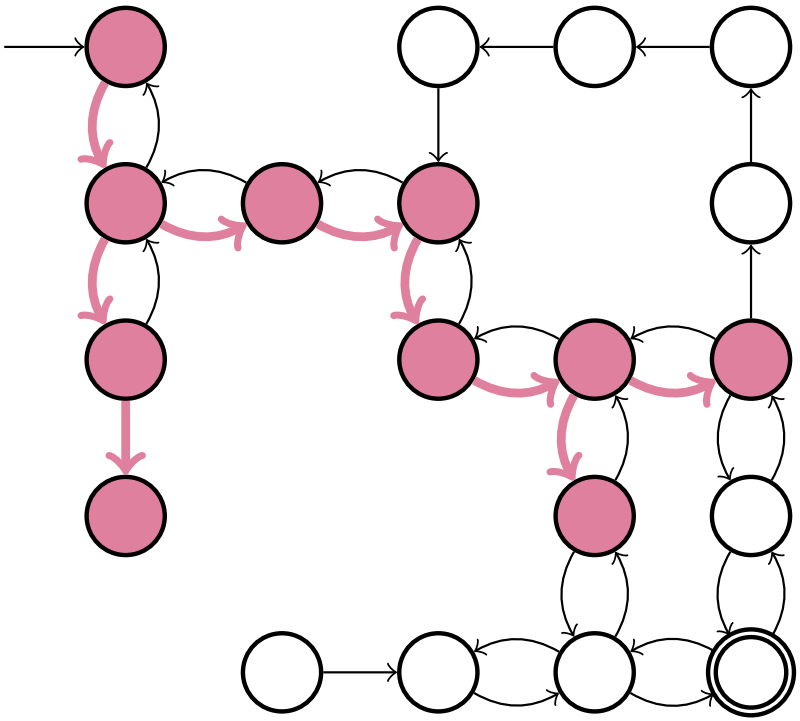

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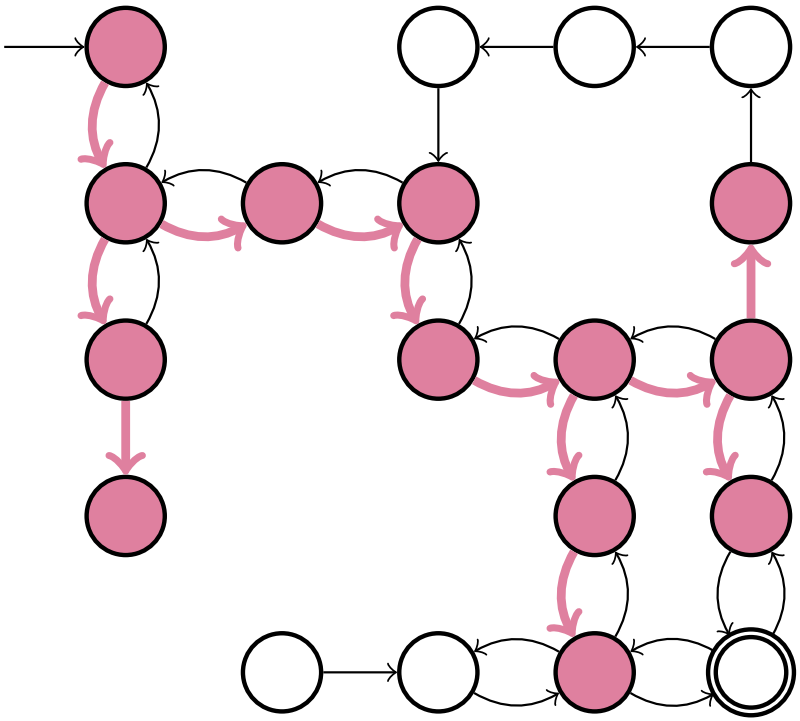

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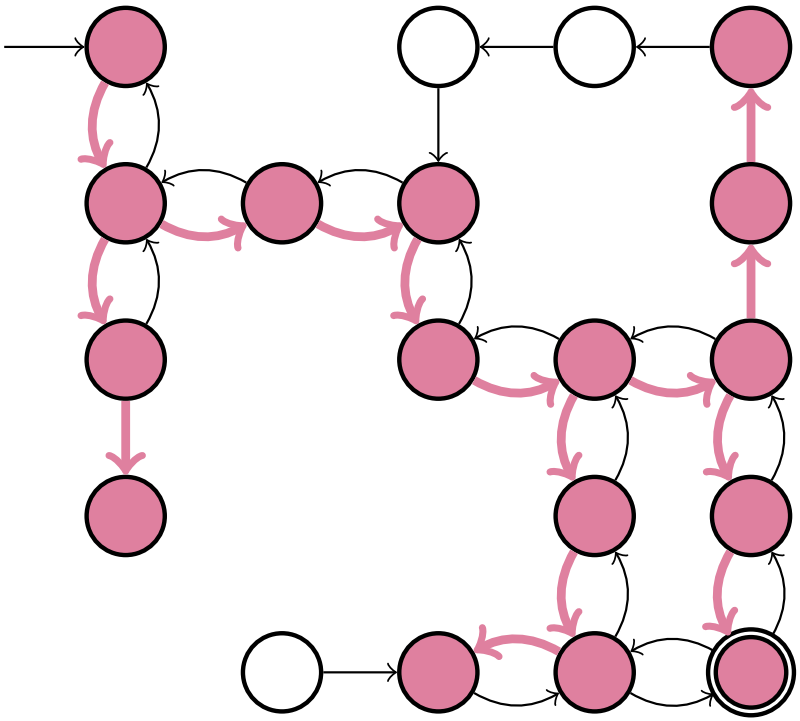

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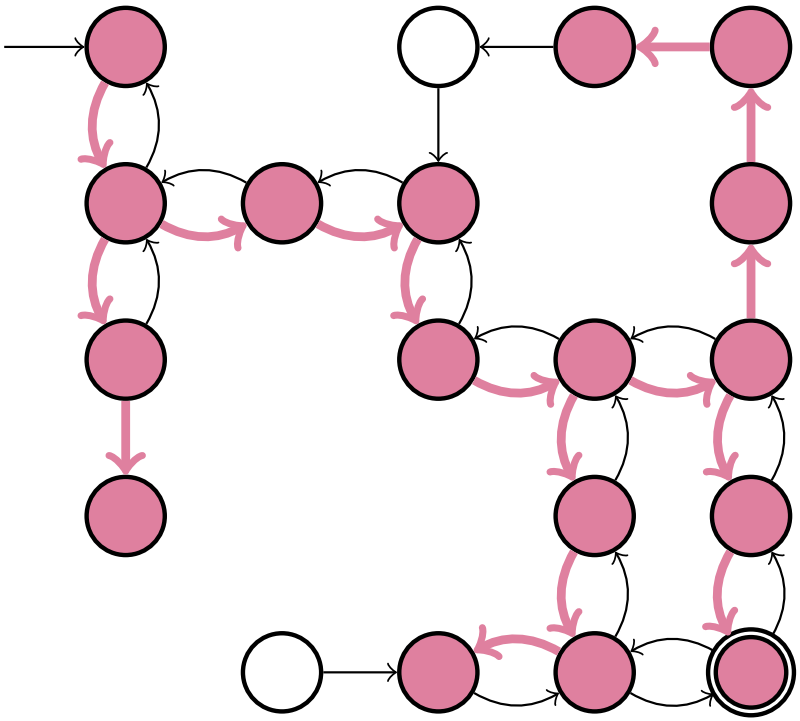

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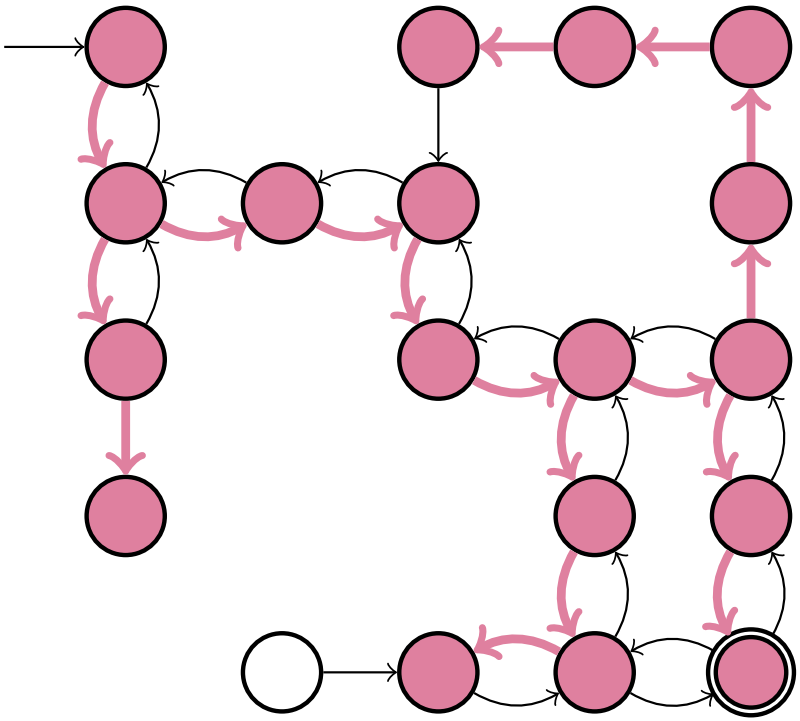

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From reachable states to the winning strategy

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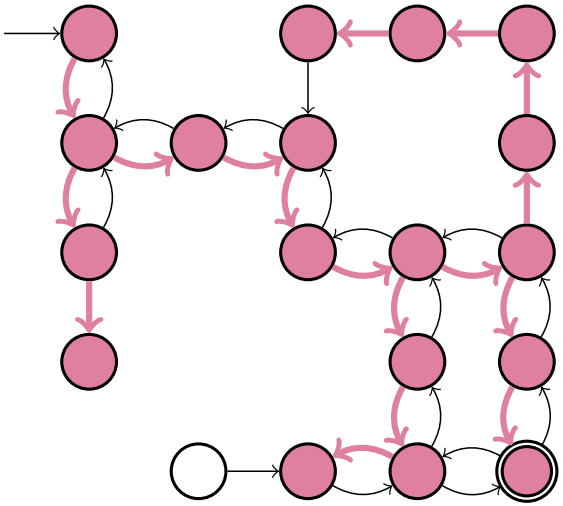
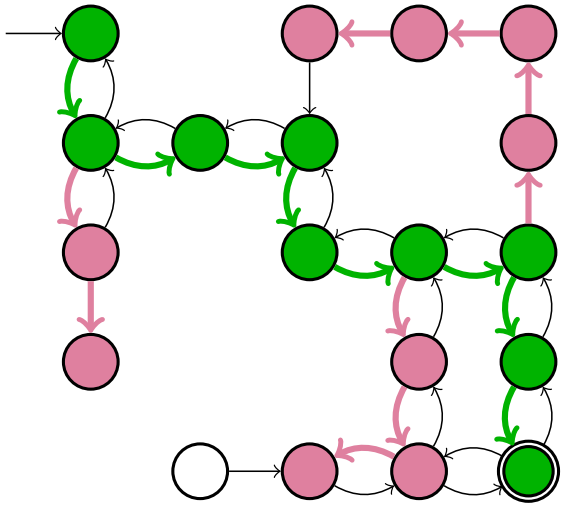
- To get the path, we classically memorize the first predecessor of each reachable state in the exploration ;
- If the goal is reachable, compute its predecessor, the predecessor of the predecessor, etc. to get the path in reverse... and the strategy.





Computation in practice

- The given fixpoint formulation is compact but not very efficient ;
- Except if we have a symbolic data structure like *Binary Decision Diagrams* (BDD) ;
- In practice, without BDDs, we could write something like :

$$
\begin{array}{l}
X \leftarrow S_0 \\
Y \leftarrow X \\
\text{while } Y \neq \emptyset : \\
\quad Y \leftarrow \text{Post}(Y) \setminus X \\
\quad X \leftarrow X \cup Y
\end{array}
$$

Reachable states : invariant

Lemma

For all $n \geq 0$, $s \in X_n$ if and only if there exists a path with $\leq n$ edges from an initial state to s.

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  - $\boxed{\Rightarrow}$ if $s \in X_{n+1}$ then either :
    - $s \in X_n$ and the induction hyp. gives a path with $\leq n \leq n+1$ edges ;
    - or $s \in \mathrm{Post}(X_n)$, so there exists $s' \in X_n$ such that $s' \to s$. By the induction hypothesis there is a path $s_0 \rightsquigarrow s'$ with $s_0 \in S_0$ and $\leq n$ edges, and thus a path $s_0 \rightsquigarrow s' \to s$ with $\leq n+1$ edges.

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  - $\boxed{\Leftarrow}$ if there is a path $s_0 \rightsquigarrow s$ with $n+1$ or less edges, and $s_0 \in S_0$:
    - either the path has 0 edges and $s \in S_0 \subseteq X_{n+1}$;
    - or the path has at least 1 edge and can be written $s_0 \rightsquigarrow s' \to s$ with n or less edges between s_0 and s'. Then by the induction hypothesis, $s' \in X_n$ and $s \in \mathrm{Post}(X_n) \subseteq X_{n+1}$.

Reachable states : termination, correctness, and soundness

Theorem

There exists $N \geq 0$ such that for all $n \geq N$, $X_n = X_N$.

At each iteration, we can only add states and S is finite.
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Theorem

Let X^* be the computed fixpoint. State s is reachable if and only if $s \in X^*$.

If s is reachable then there is a finite path with $n \geq 0$ edges from S_0 to s.
And if $s \in X^*$ then there $s \in X_N$, by the above theorem.

Winning states

- A state is winning if there is a winning strategy from it ;
- We have computed (some) winning states by backtracking from the goal using reachable states and their predecessors ;
- We could compute them more directly ;
- What makes a winning state ?

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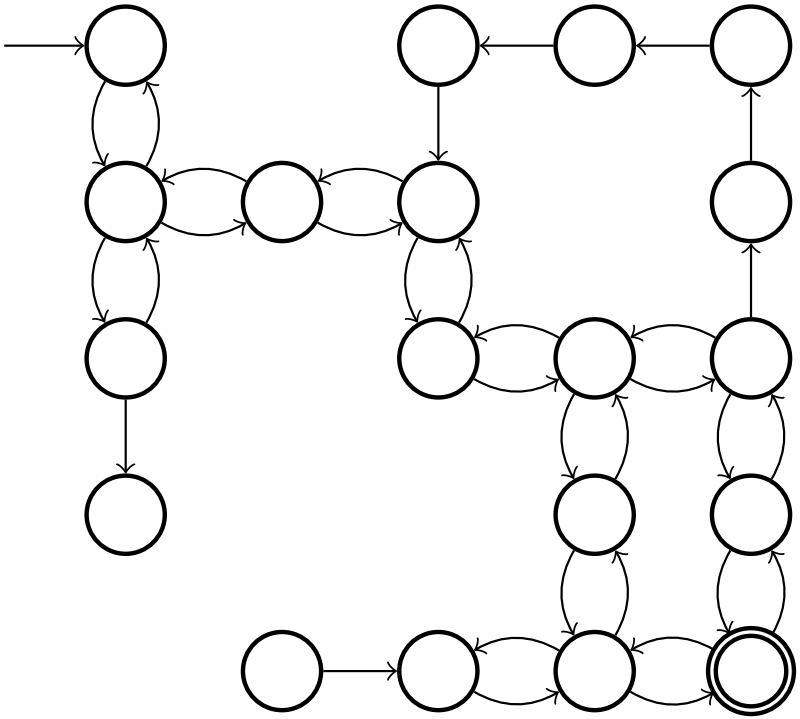
- What is then a winning strategy ? Going to a winning successor !

Winning (co-reachable) states



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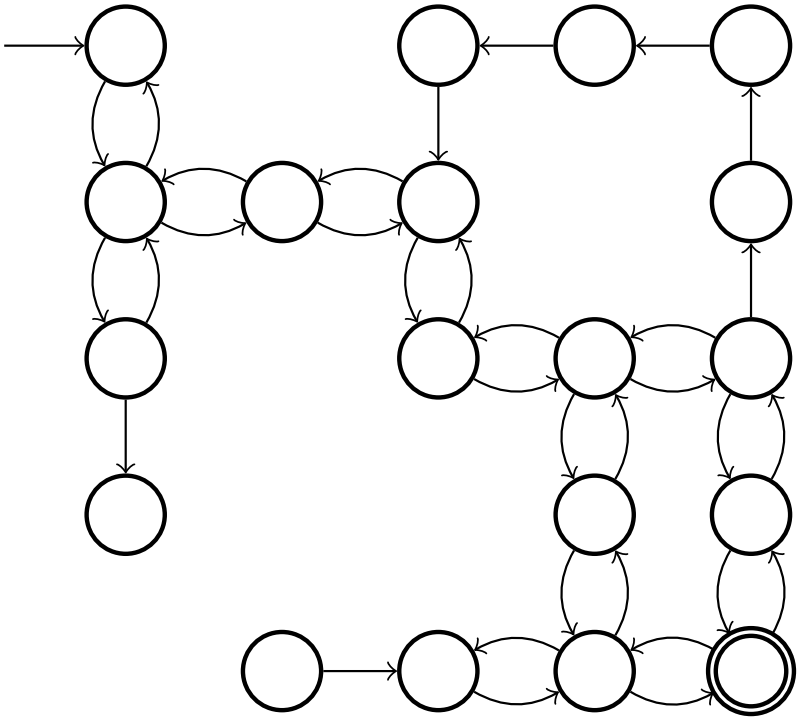

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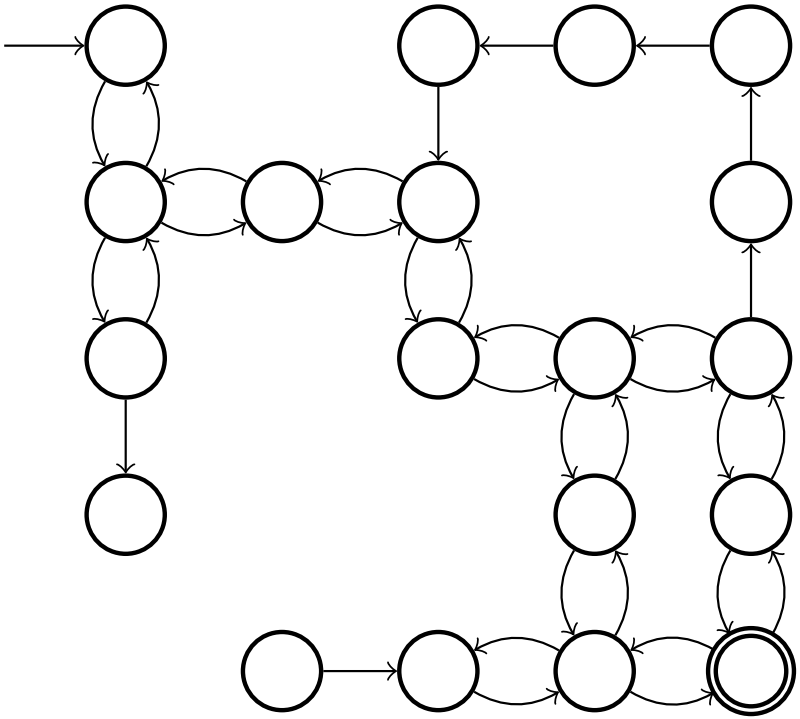

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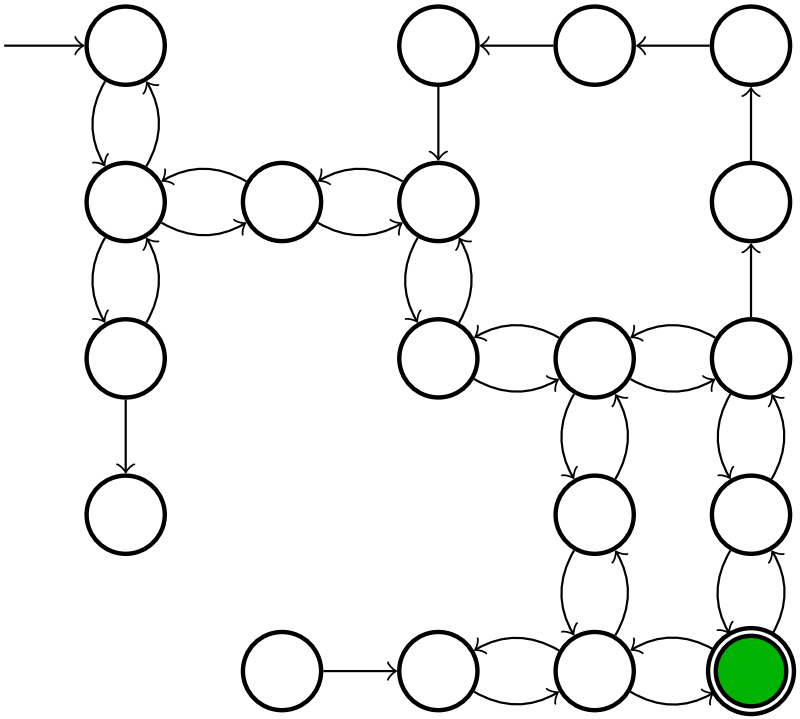

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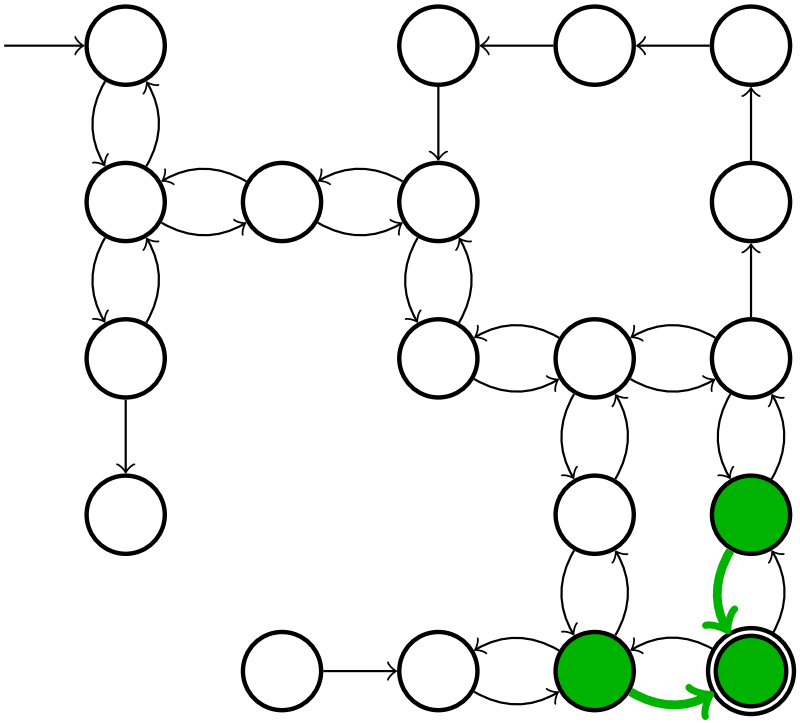

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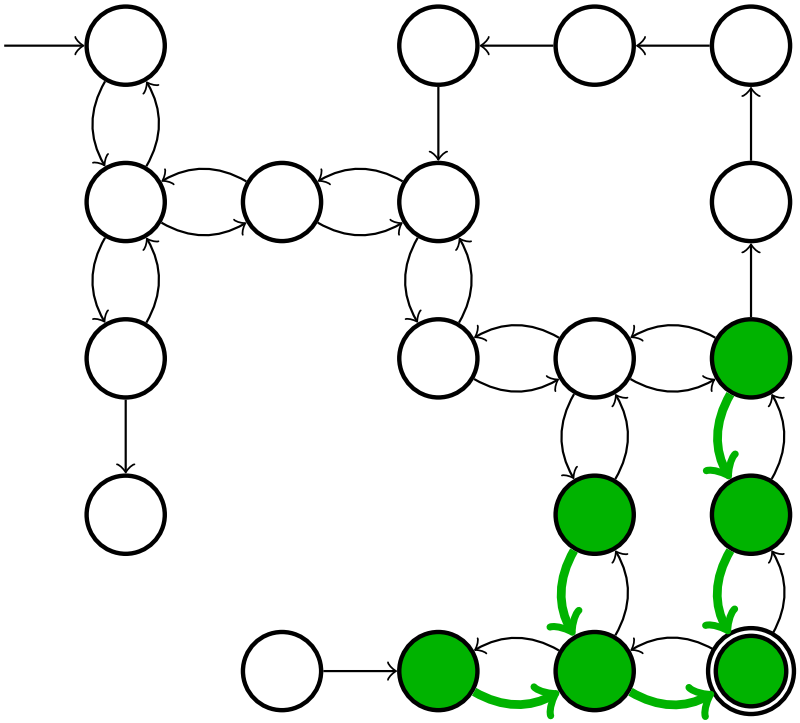

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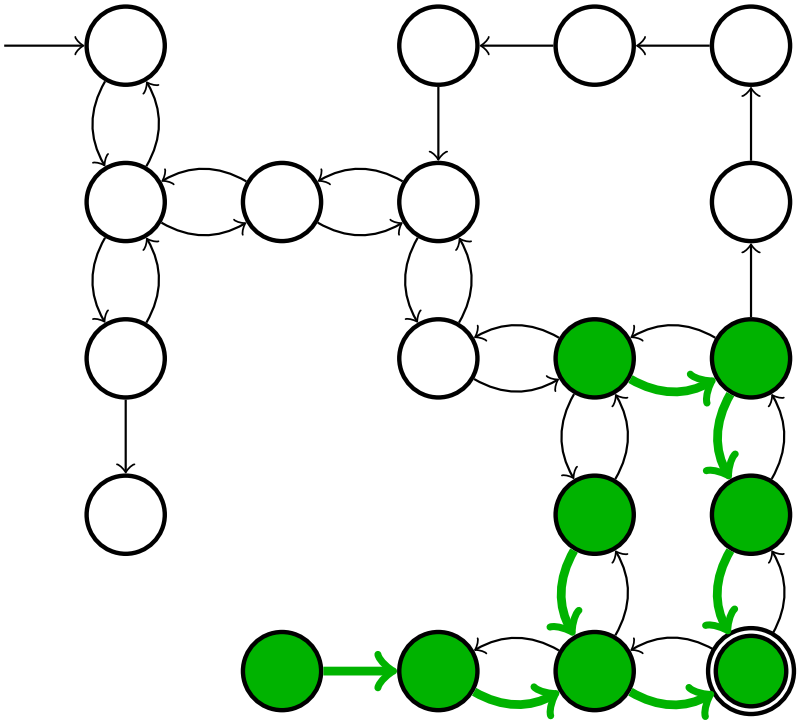

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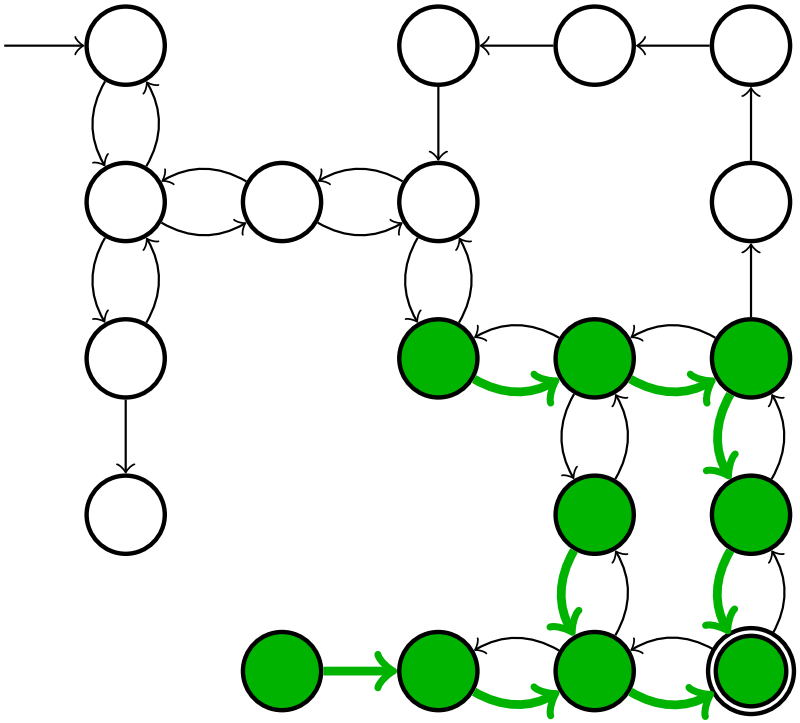
Winning (co-reachable) states



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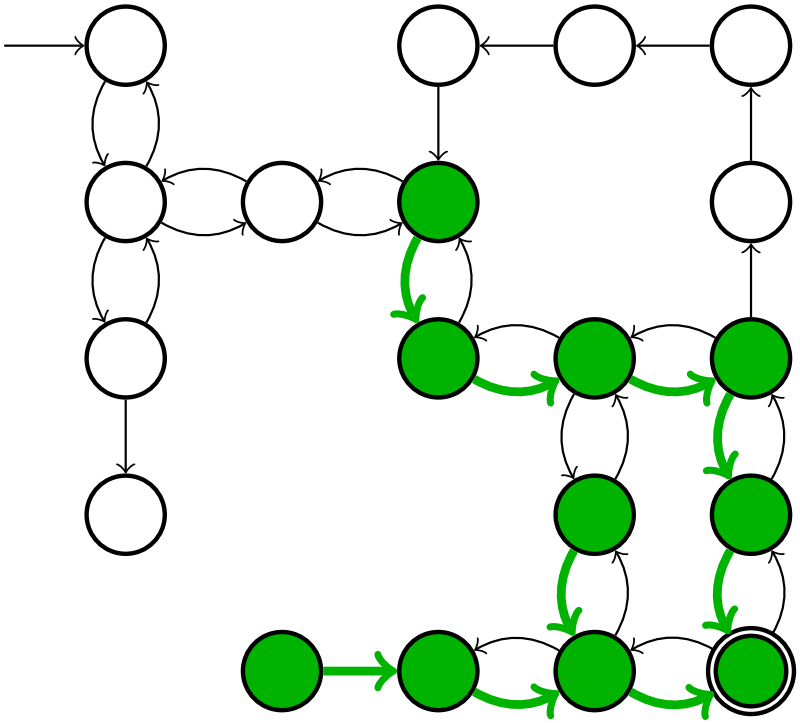

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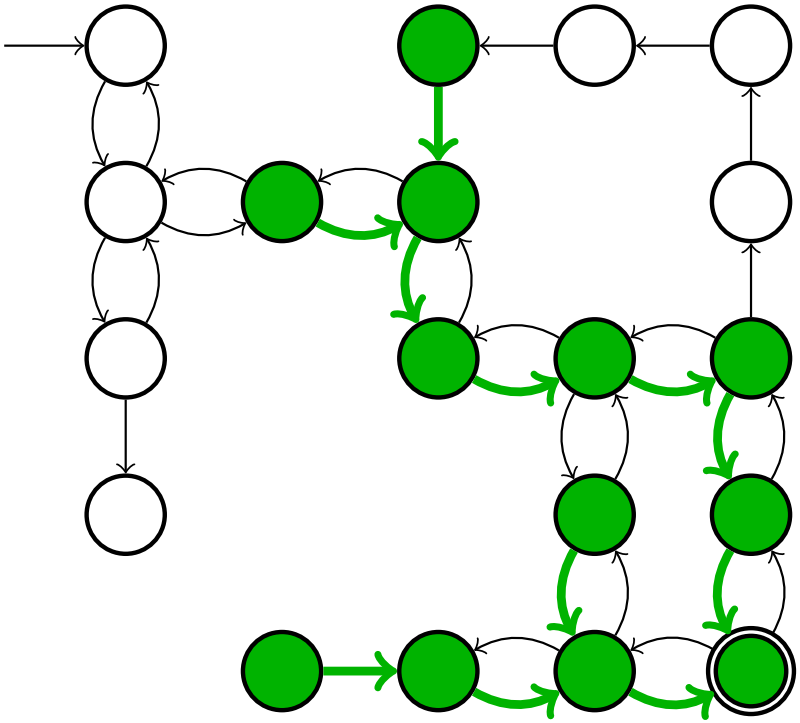

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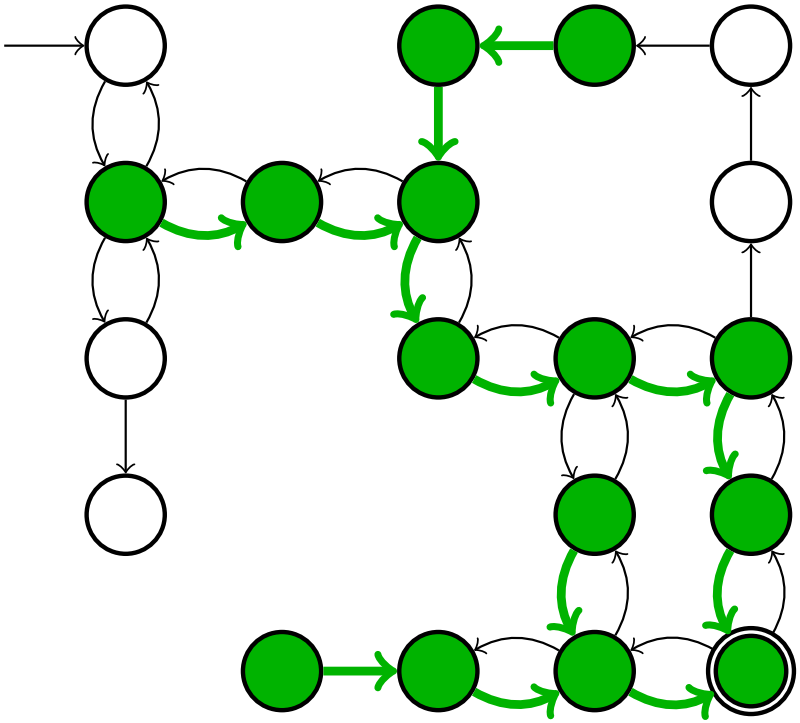

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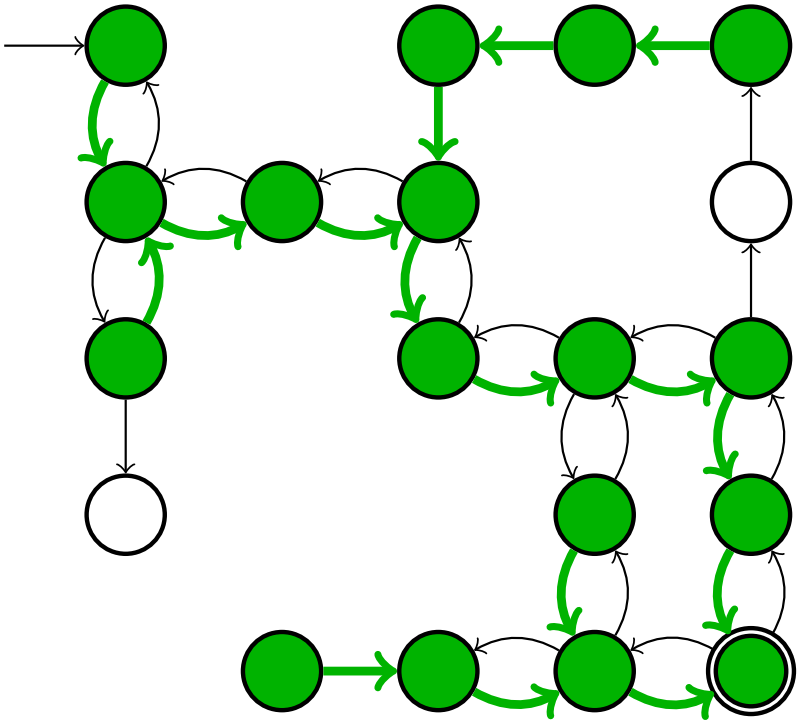

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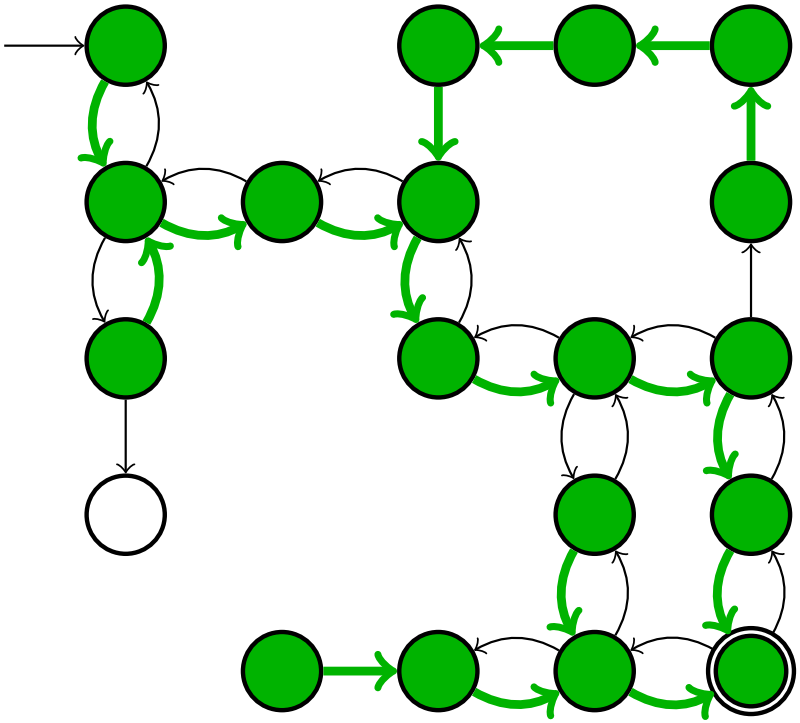

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Computing strategies

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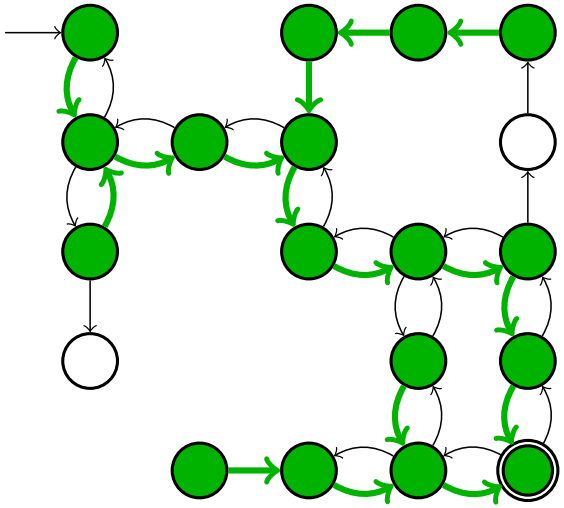

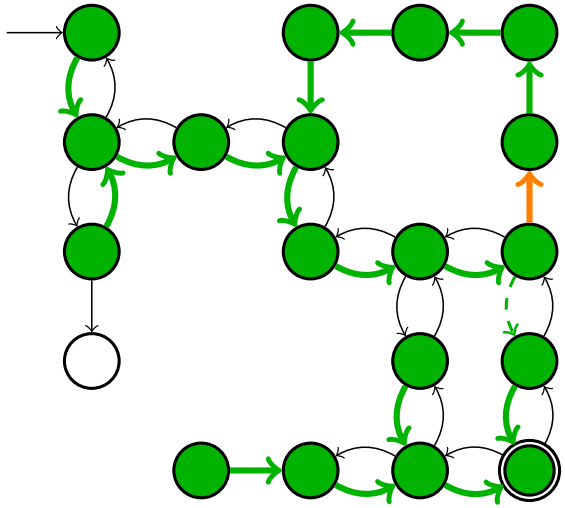

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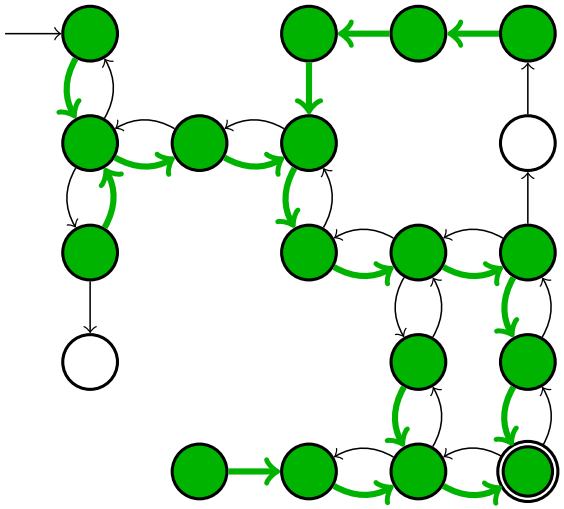

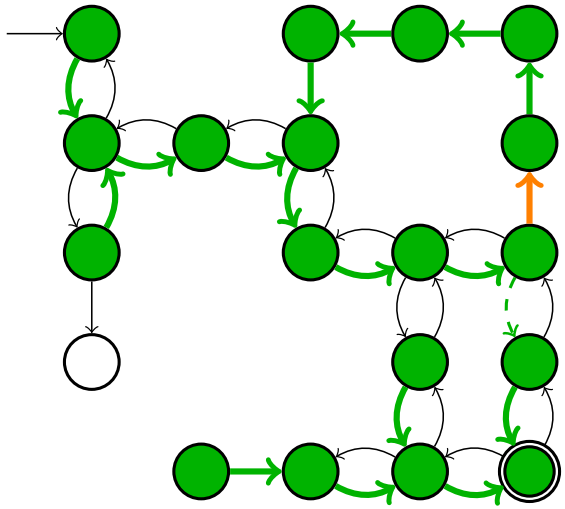

- We need to ensure progress :

Lemma

For all $n \geq 0$, $s \in X_n$ if and only if there exists a path with $\leq n$ edges from s state to a goal state.

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- Let $\widetilde{\mathrm{Pre}}(Z) = \{s | s \to s' \Rightarrow s' \in Z\}$, we have :

$$\begin{cases} Y_0 = G \\ Y_{n+1} = Y_n \cap \widetilde{\mathrm{Pre}}(Y_n) \end{cases}$$

Safety game

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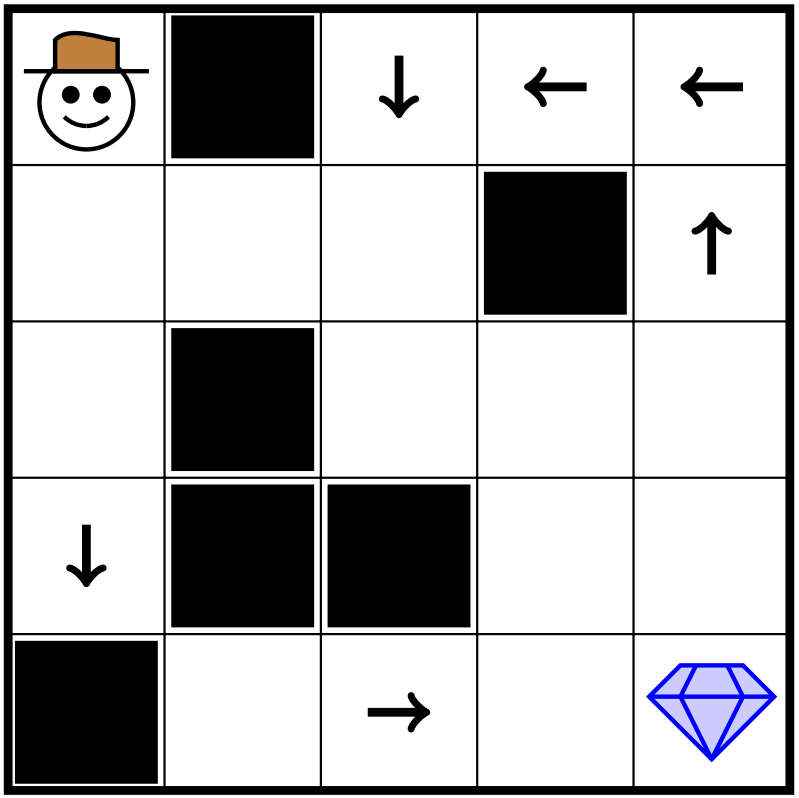

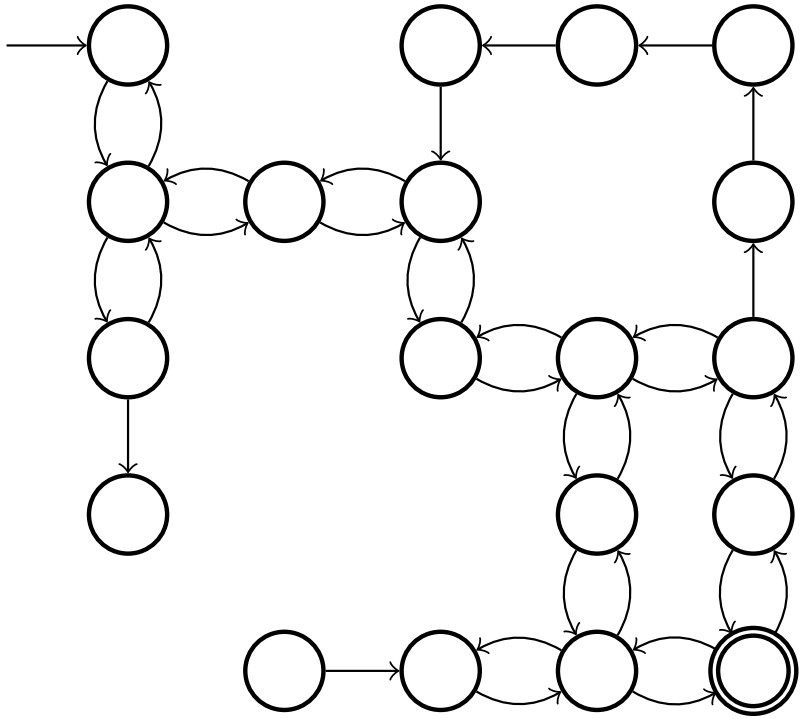

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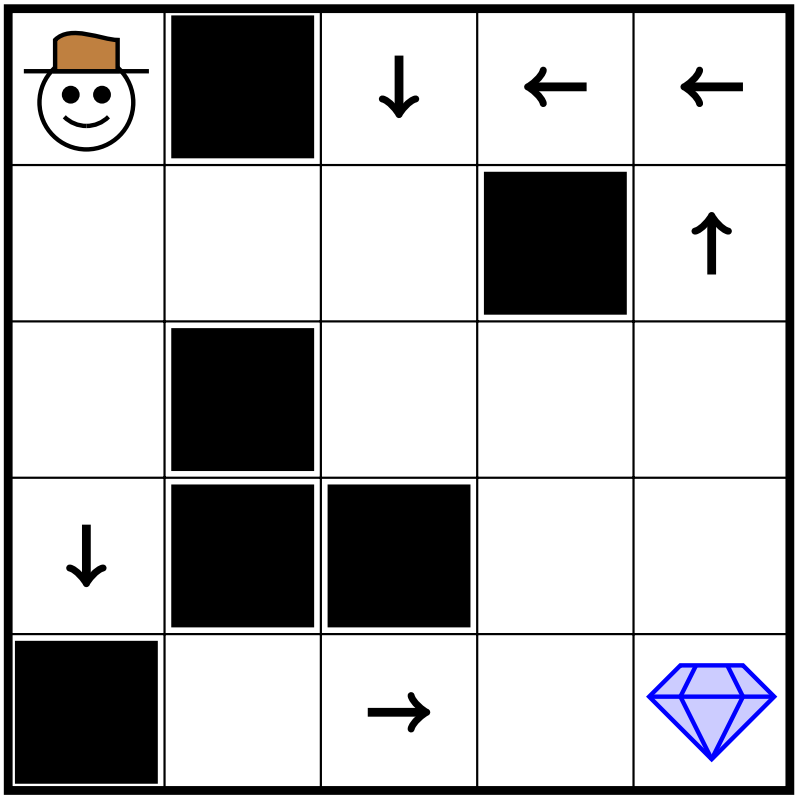

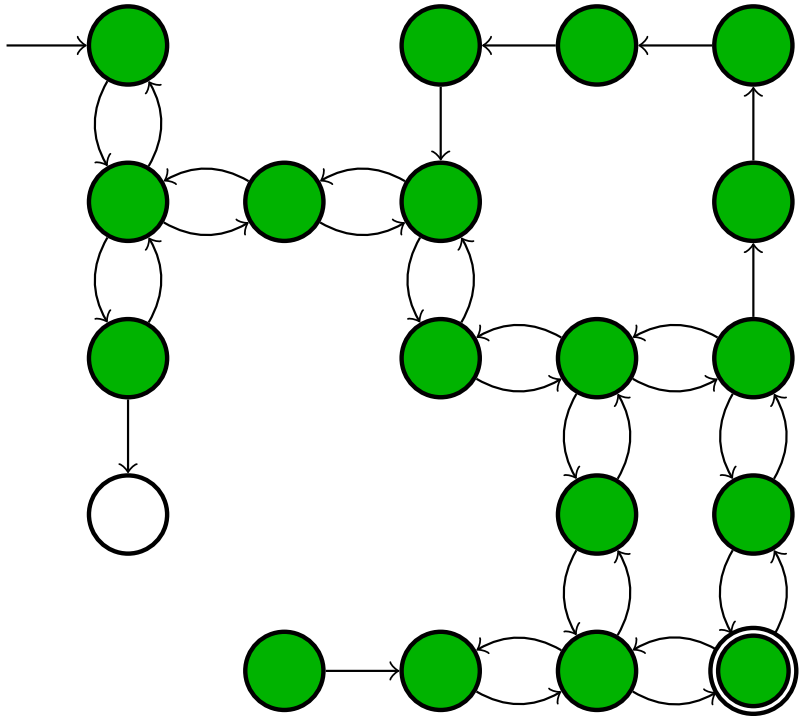

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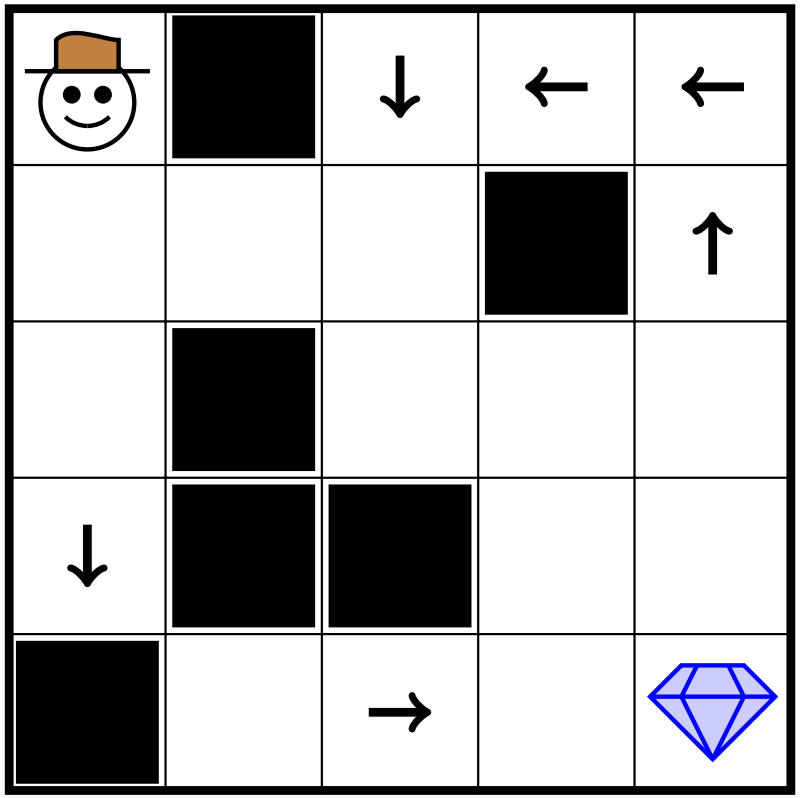

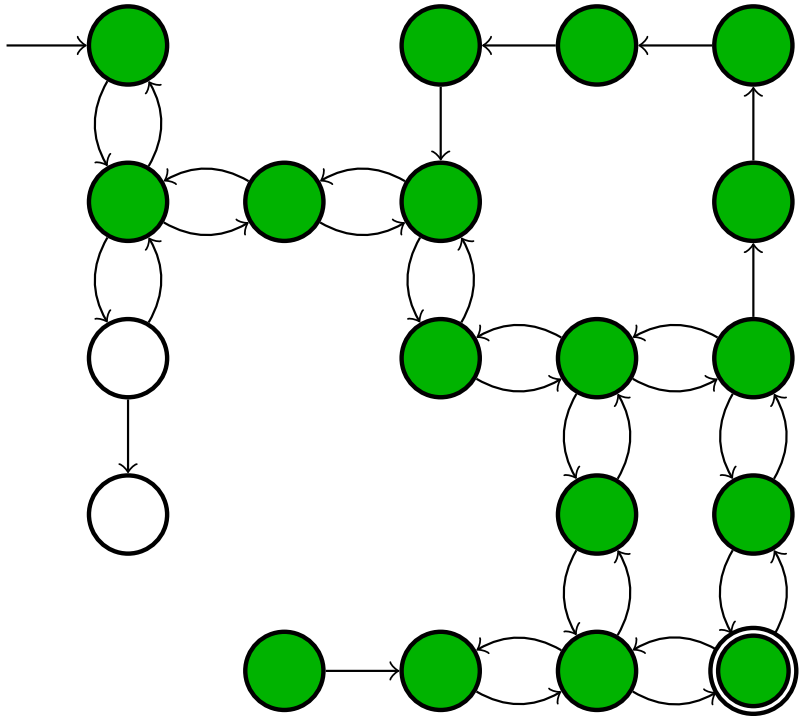

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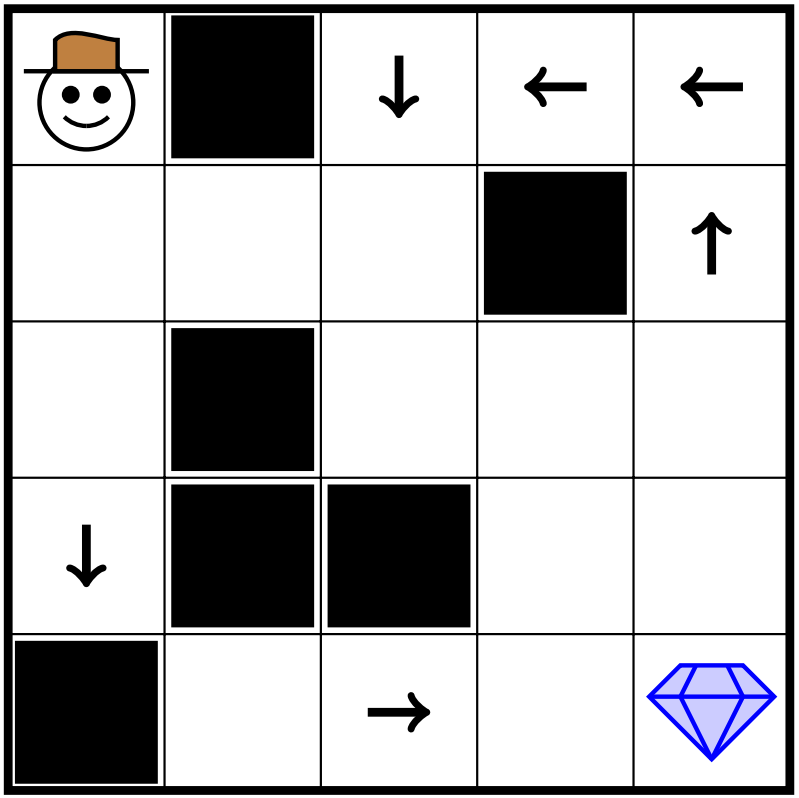

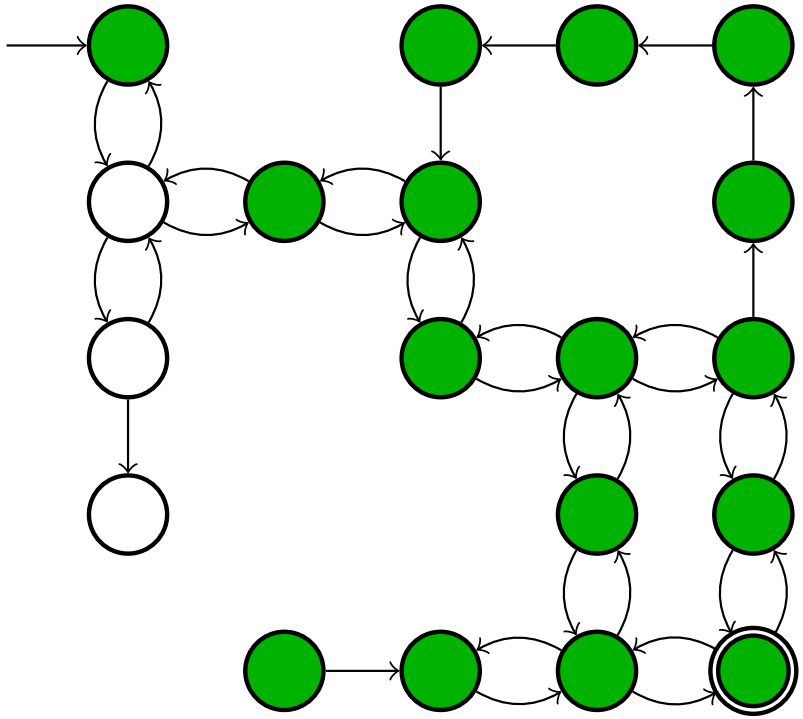

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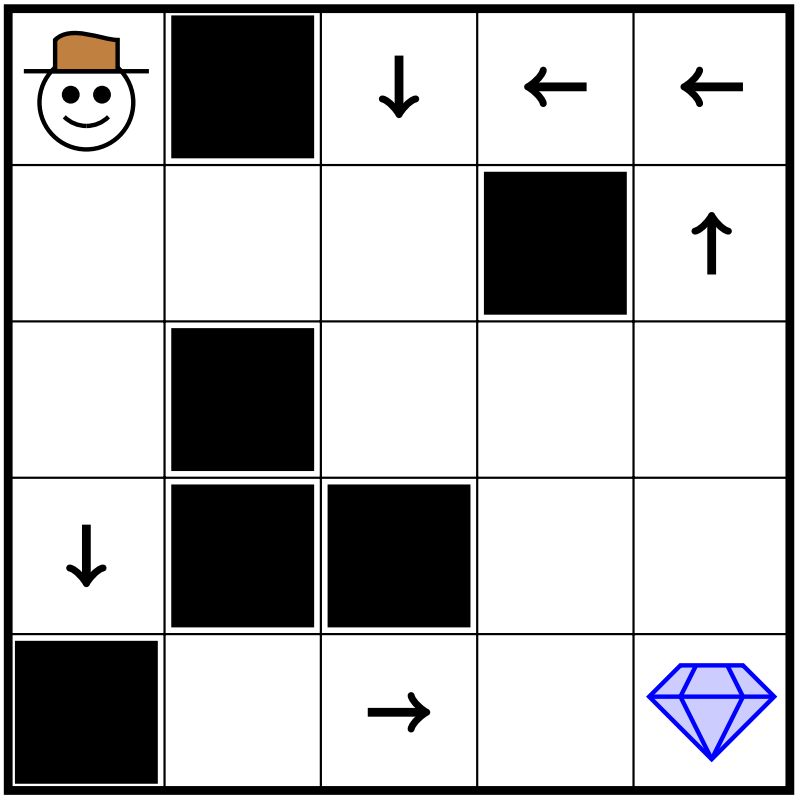

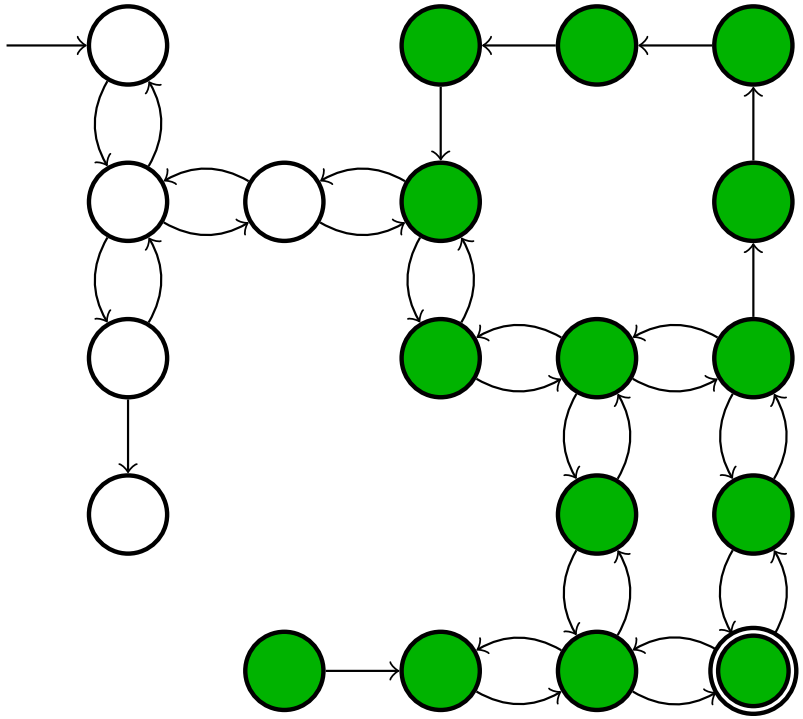

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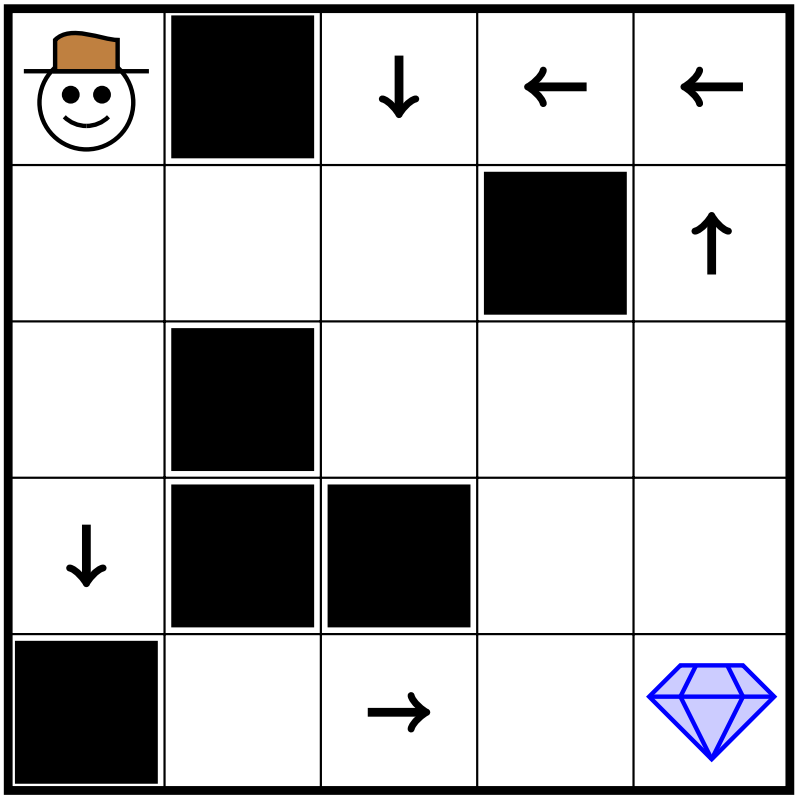

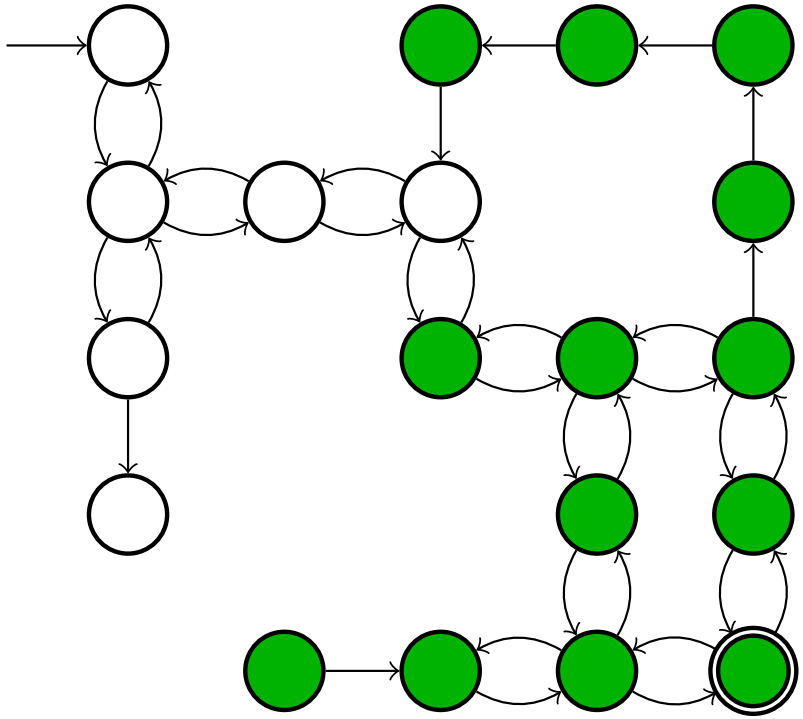

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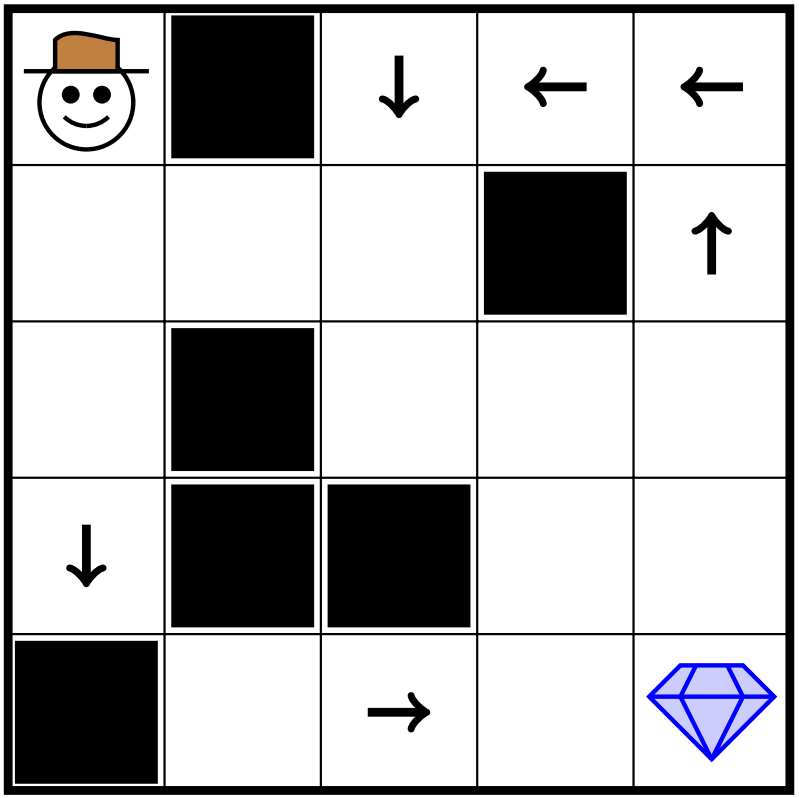

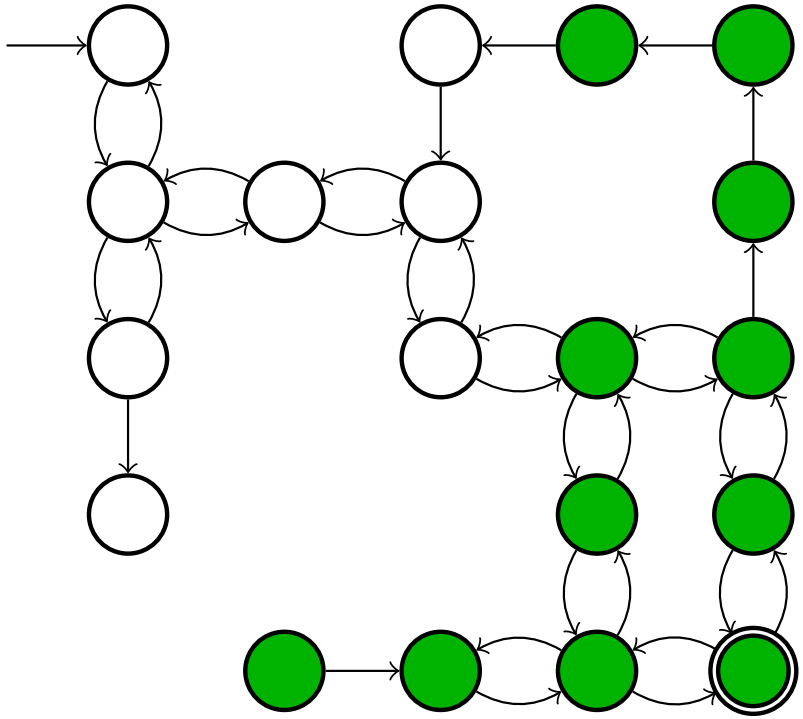

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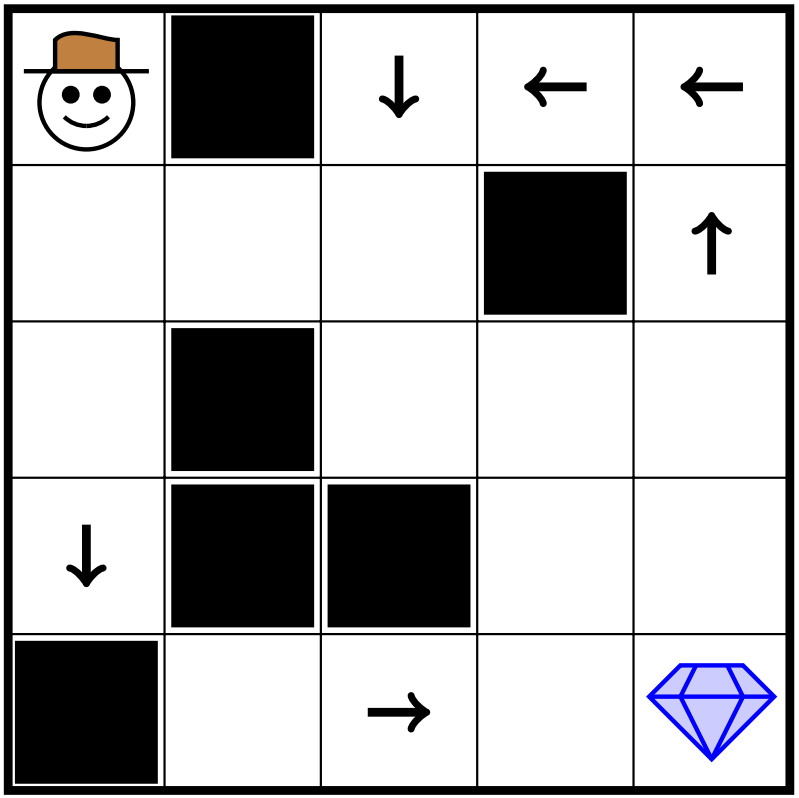

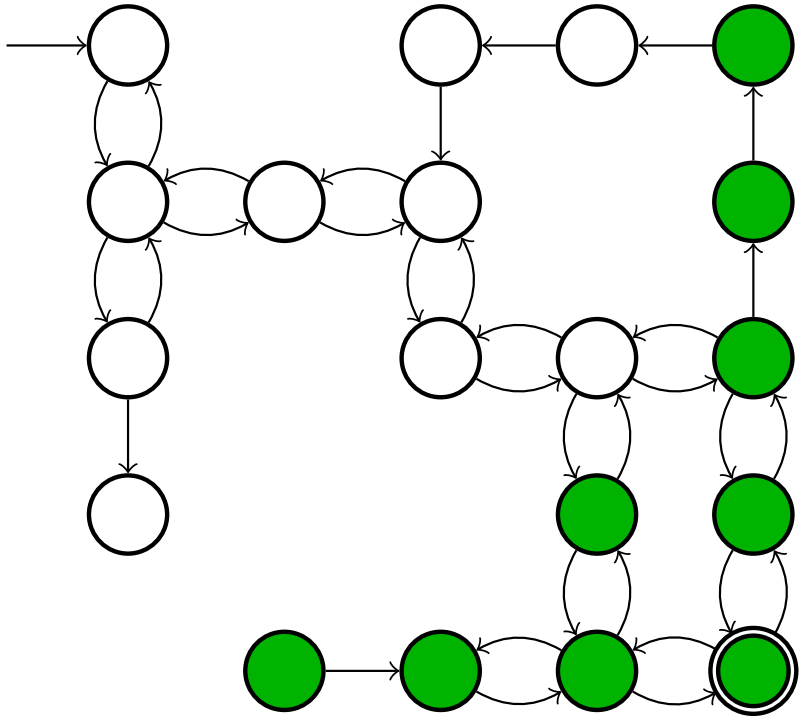

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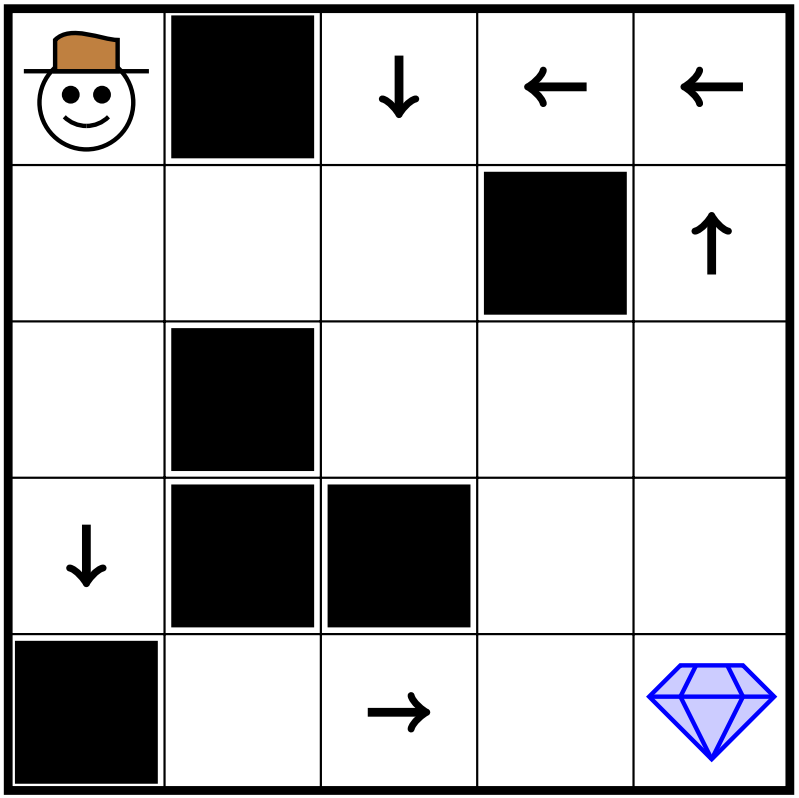

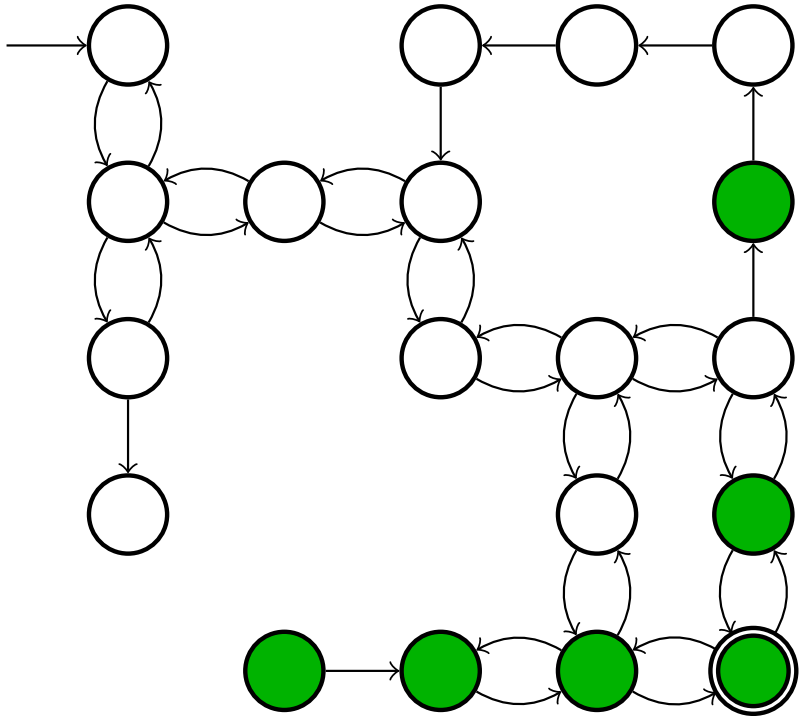

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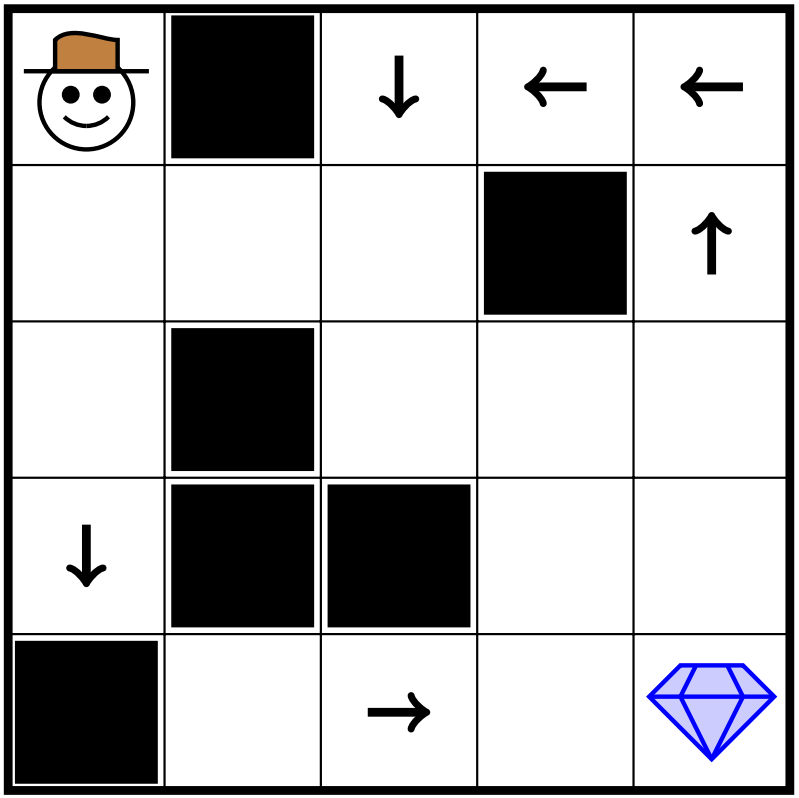

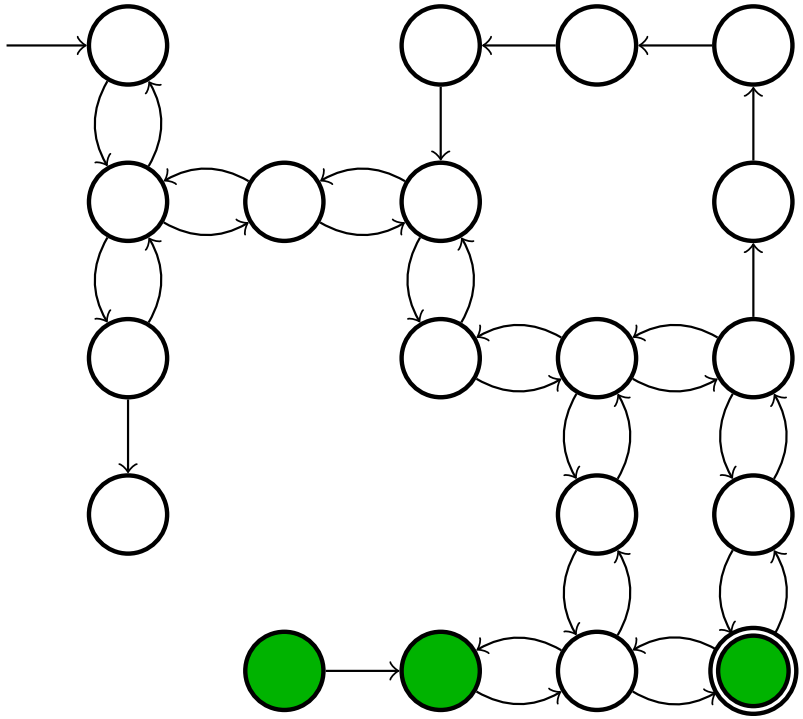

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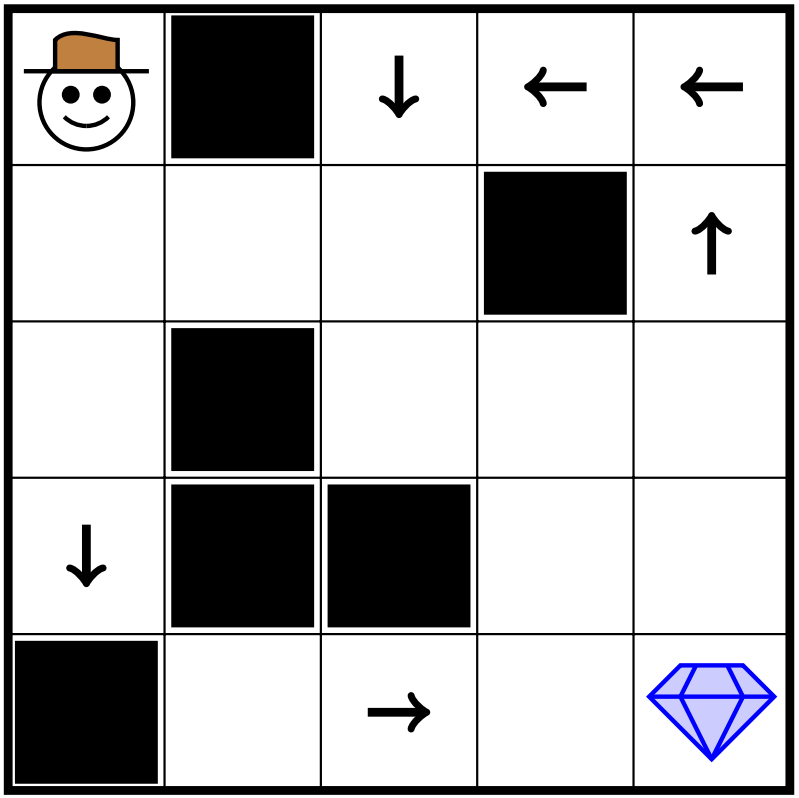

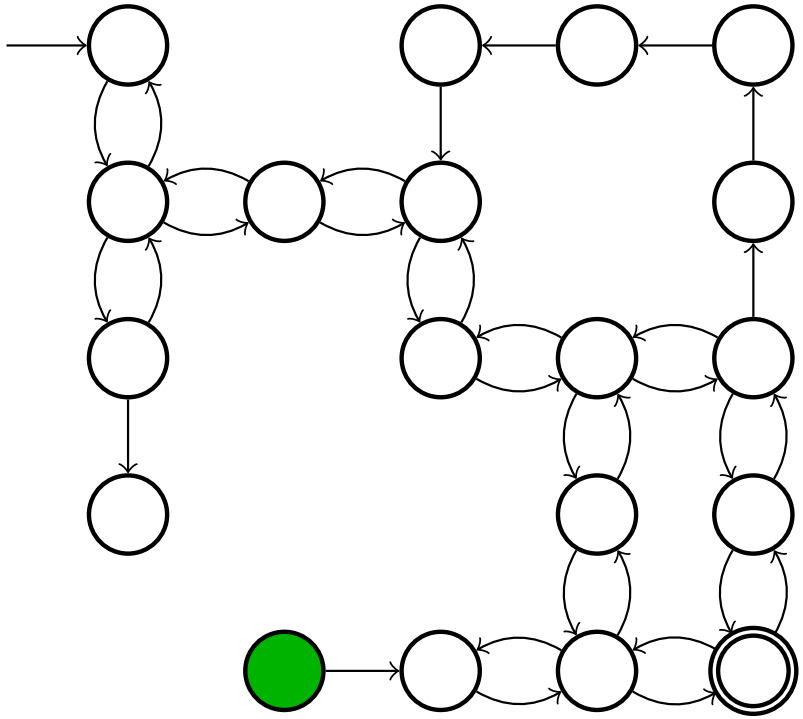

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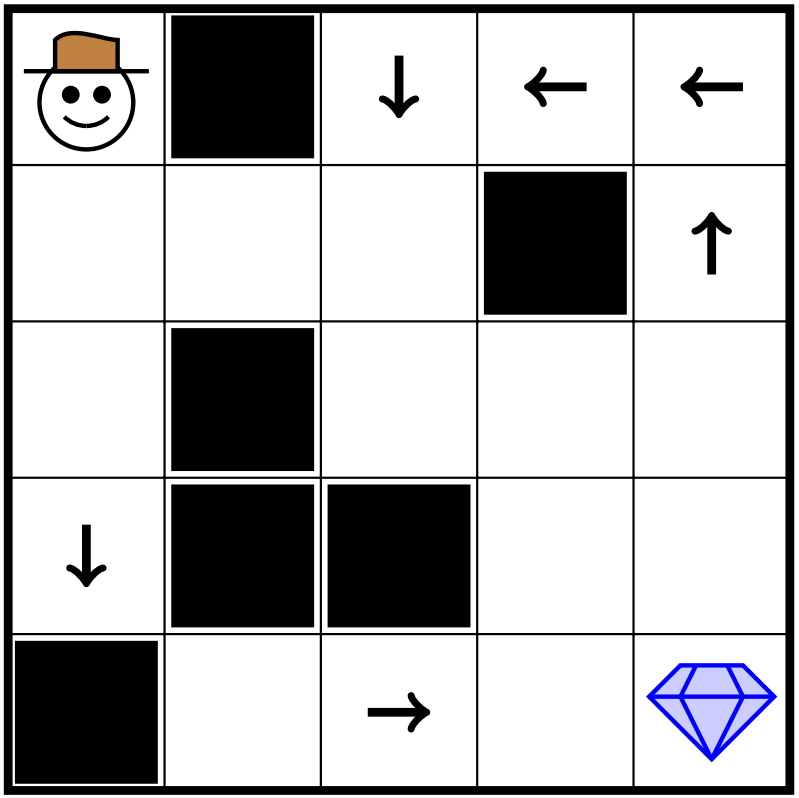
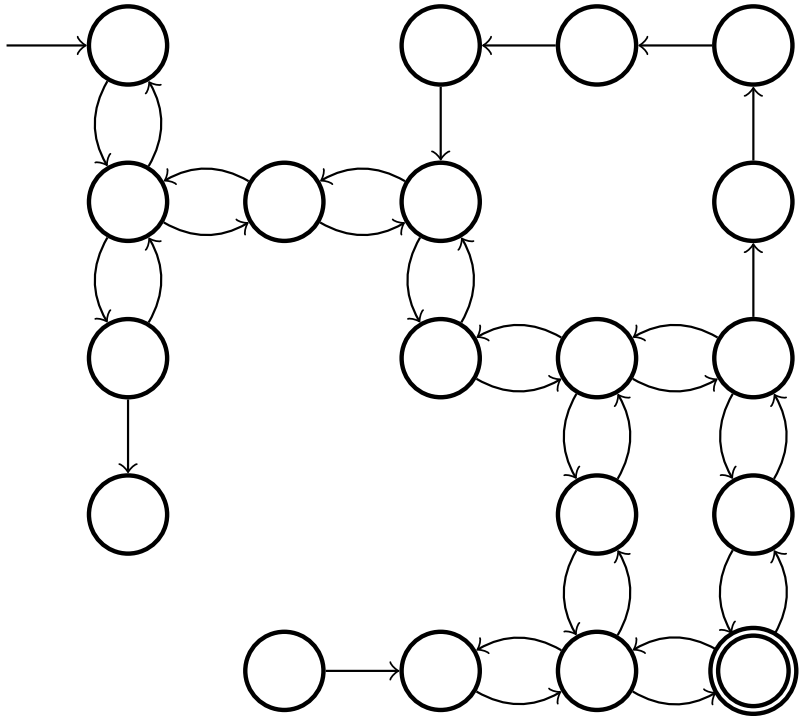
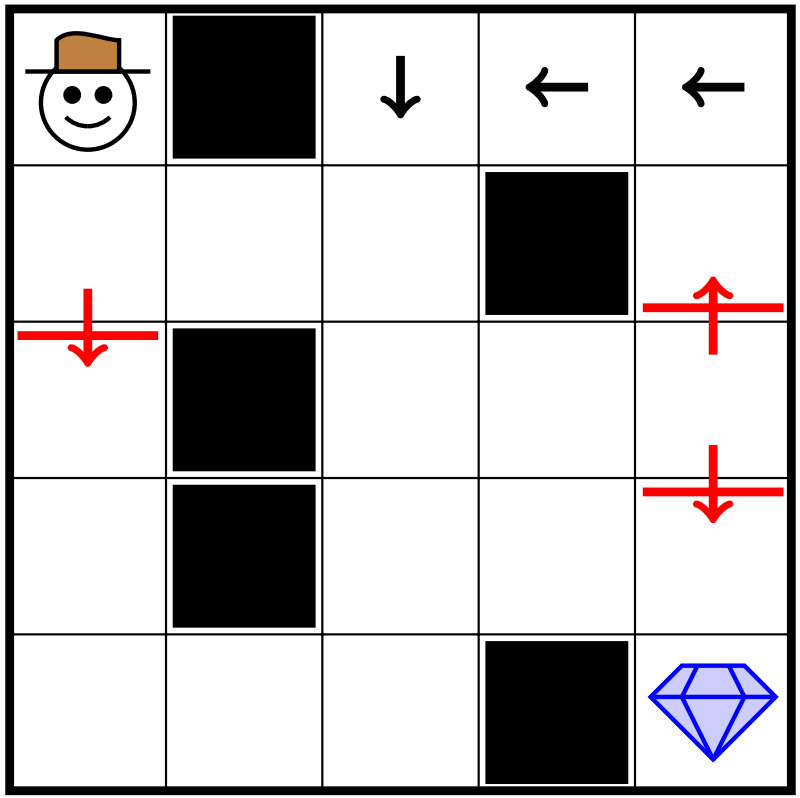




Two-player games

Two-player reachability games



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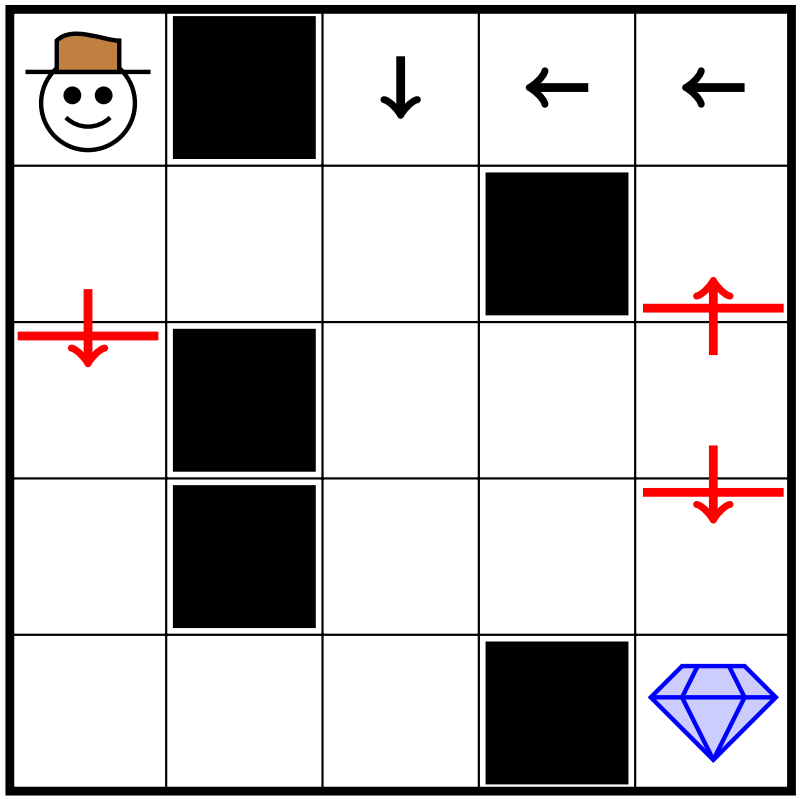

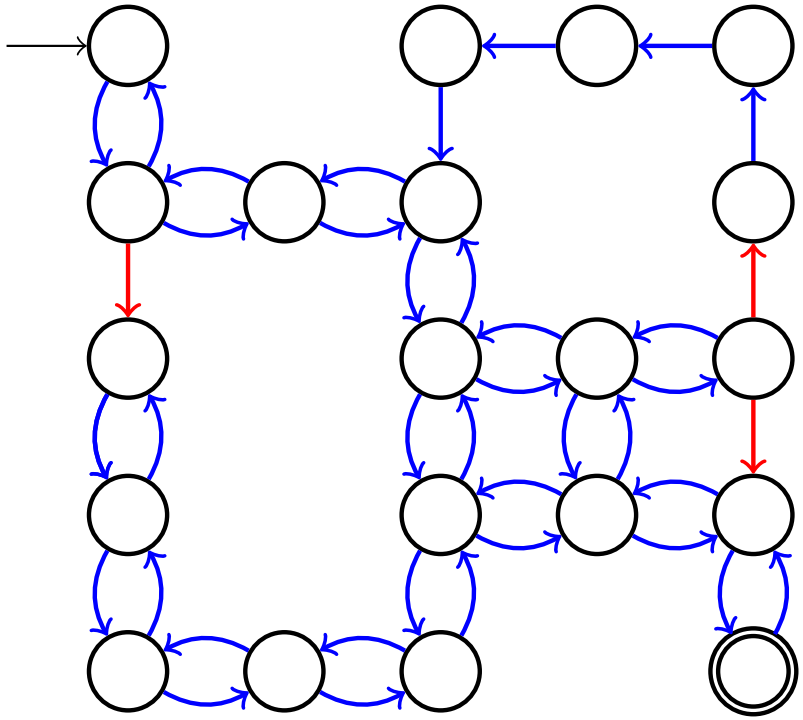

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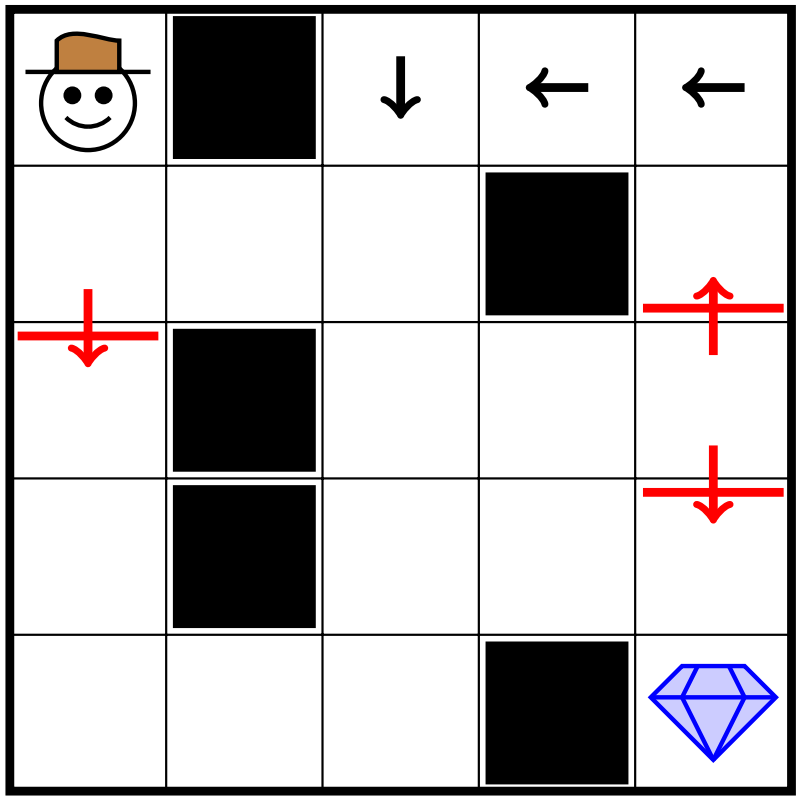

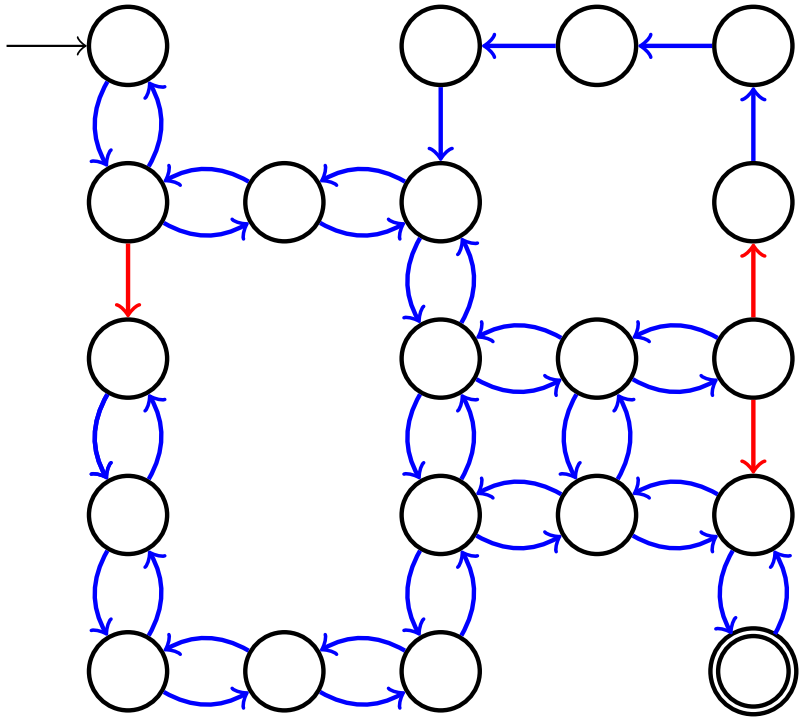

Two players : controller ( ) and environment (traps)

Two-player reachability games : playing and winning

- Strategies for each player allow them only to chose among their own actions ;

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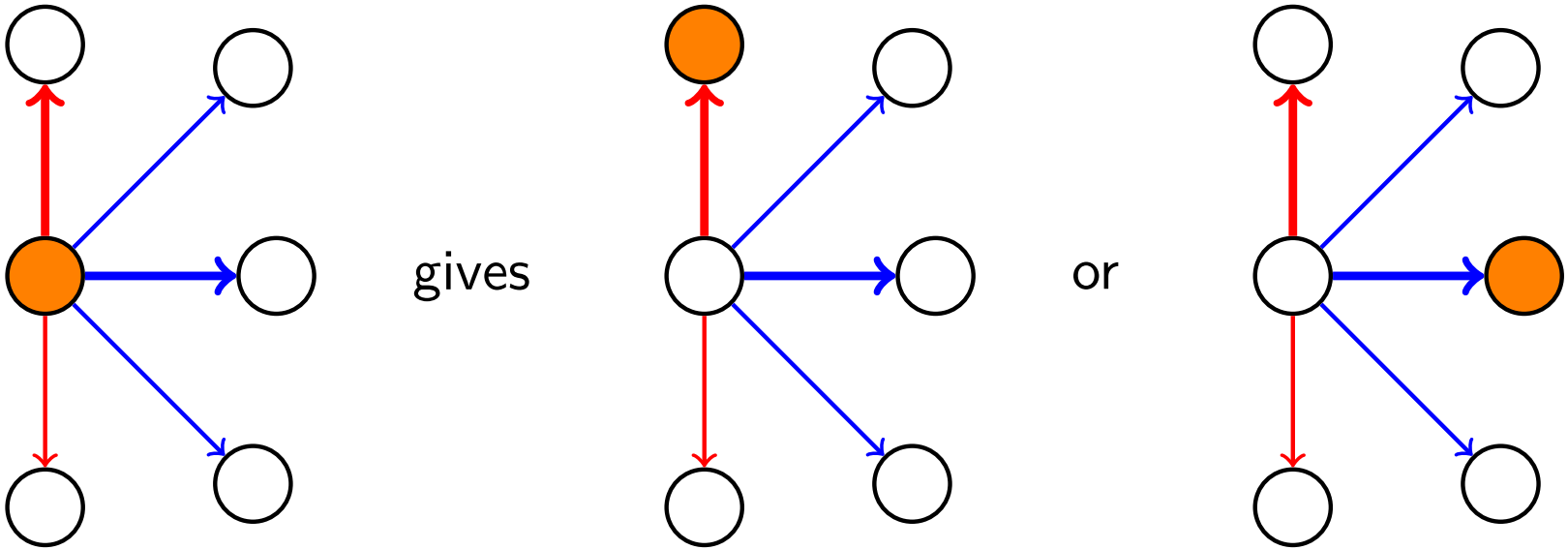

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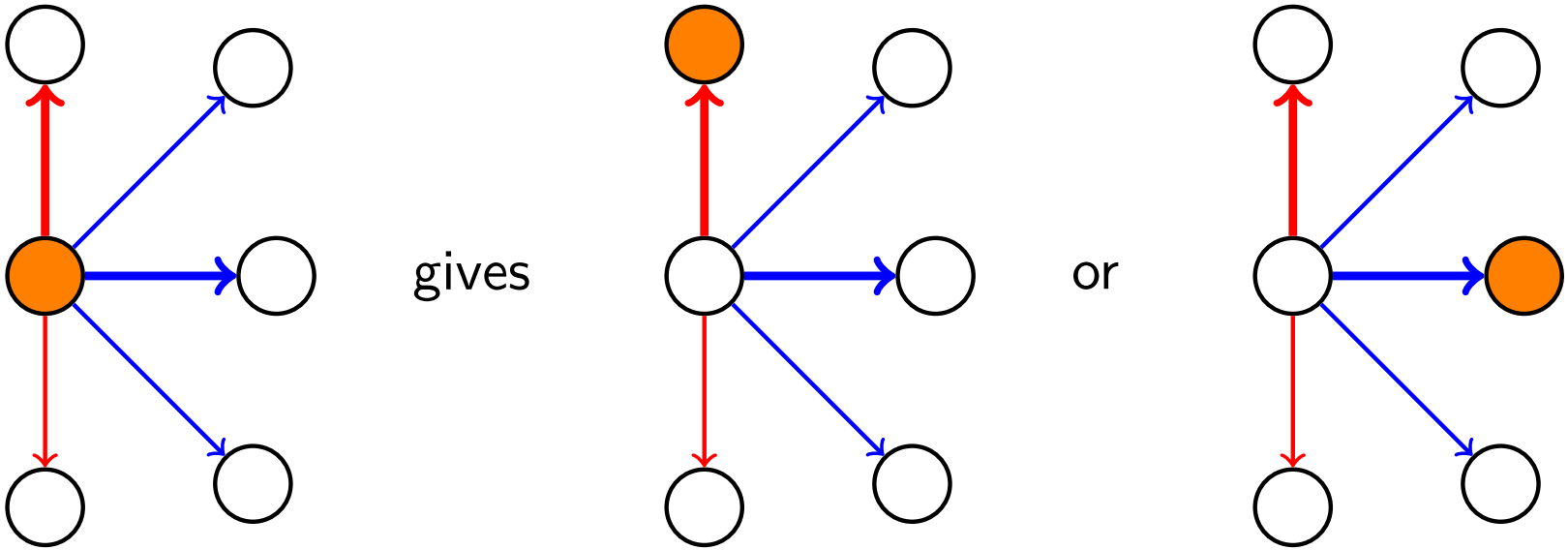

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Two-player reachability games : playing and winning

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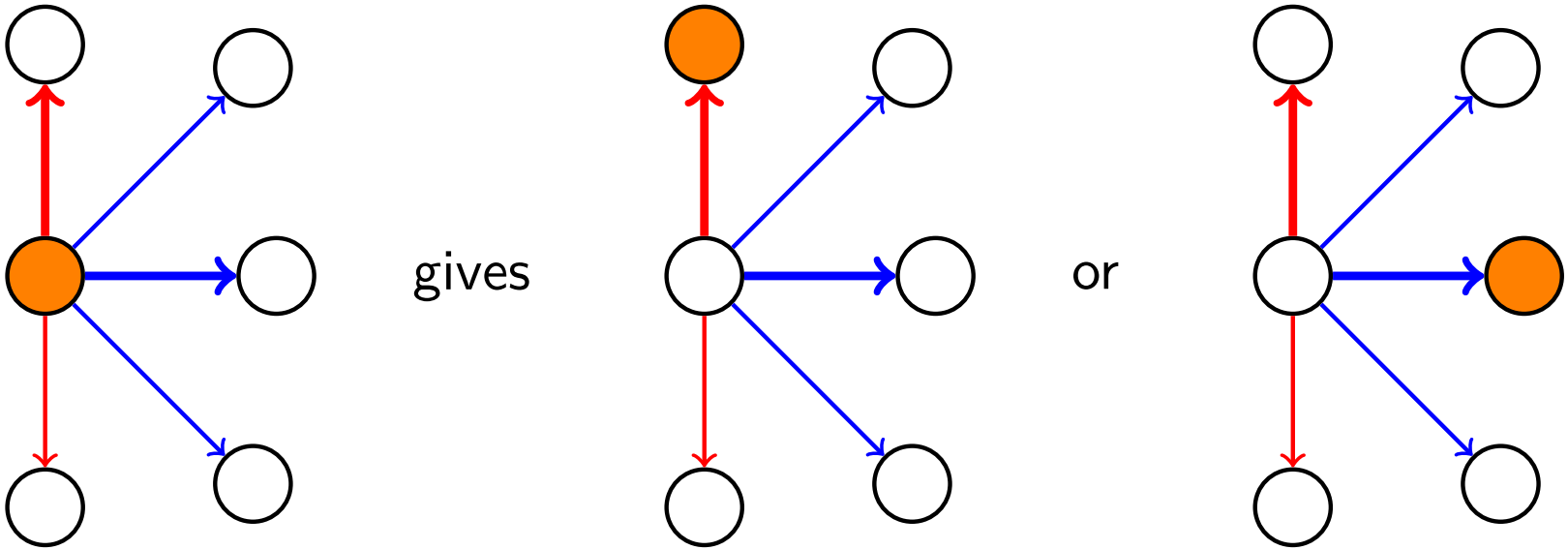
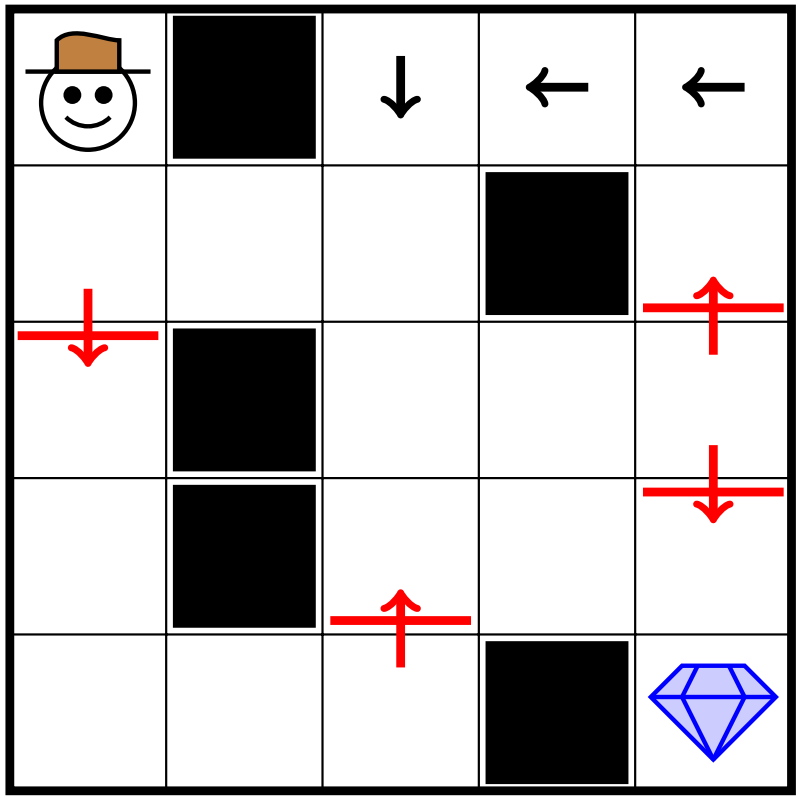


- Worst case : the environment can always be faster if they want to ;
- A (controller) strategy is winning if for all strategies of the environment, all corresponding maximal plays intersect the goal.

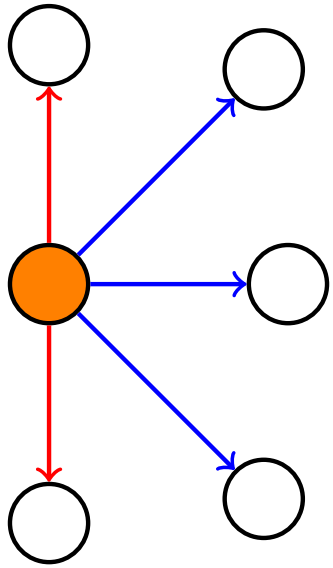
Two-player reachability games



The environment can prevent us from winning :

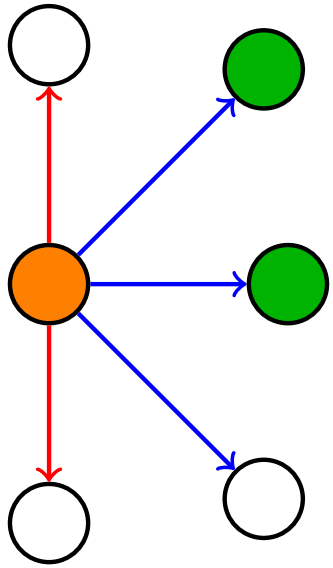
1. By doing something ;
2. By NOT doing something.

Winning states



What makes a winning state ?

Winning states



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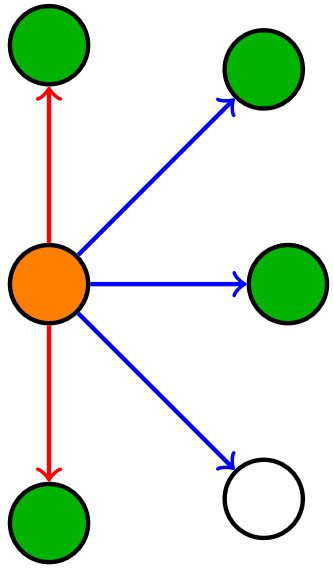

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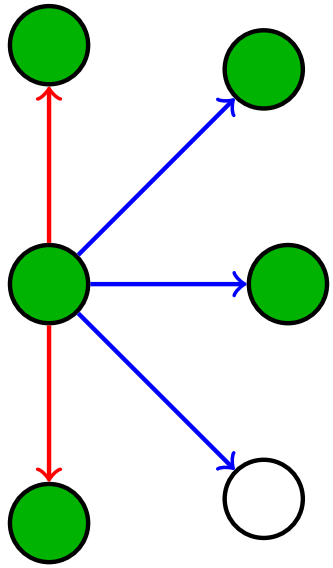

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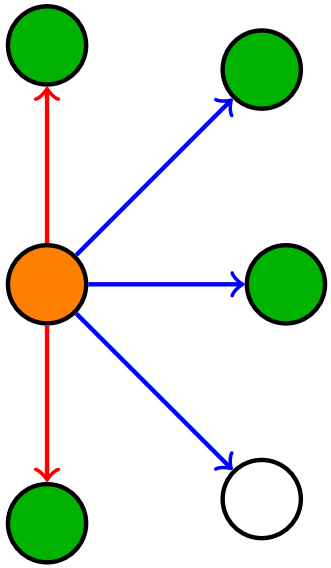

How to build the winning strategy ?

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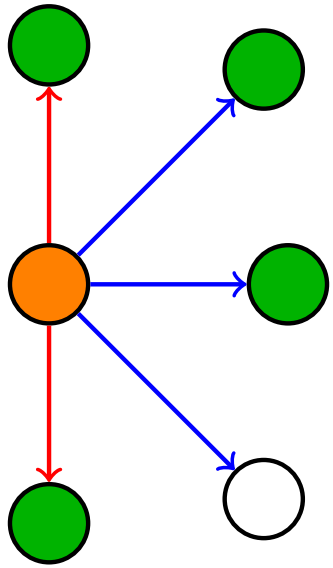

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When a state is first computed to be winning, choose one of its controllable winning successors.

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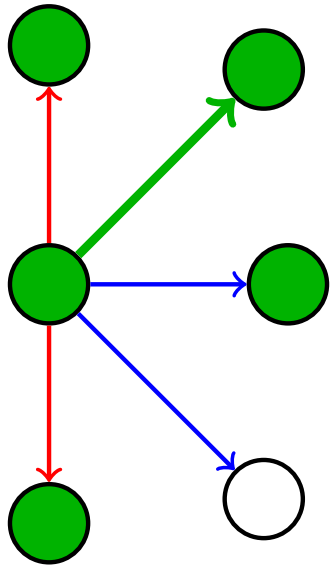

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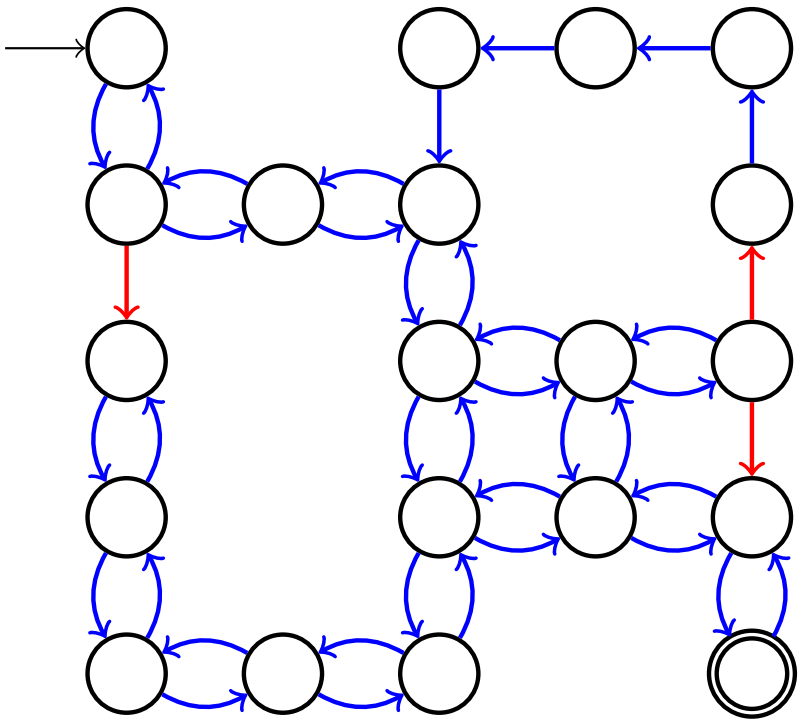

Extend the co-reachability algorithm :

$$X_0 = G$$
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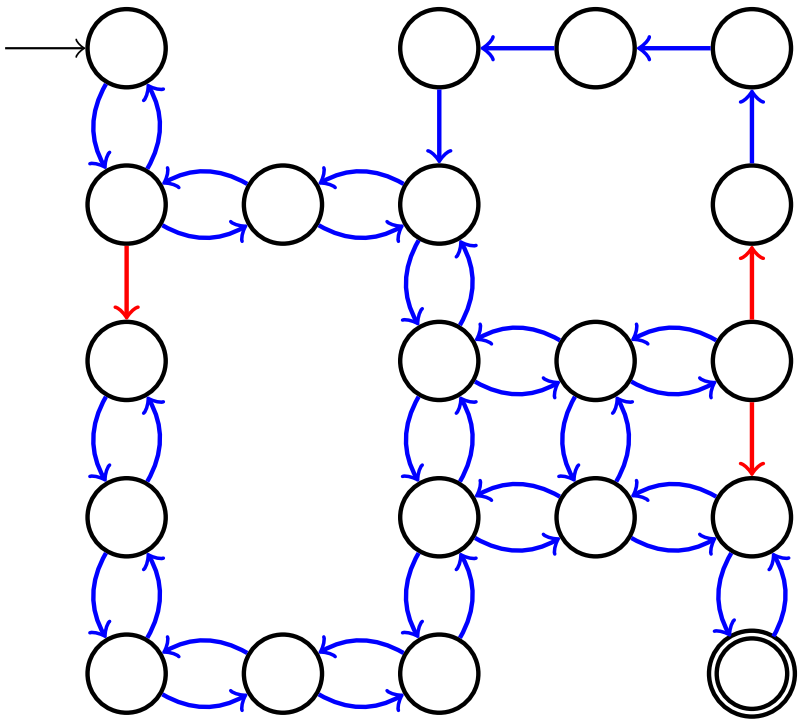
With :

$$\pi(X) =$$

Computing winning states



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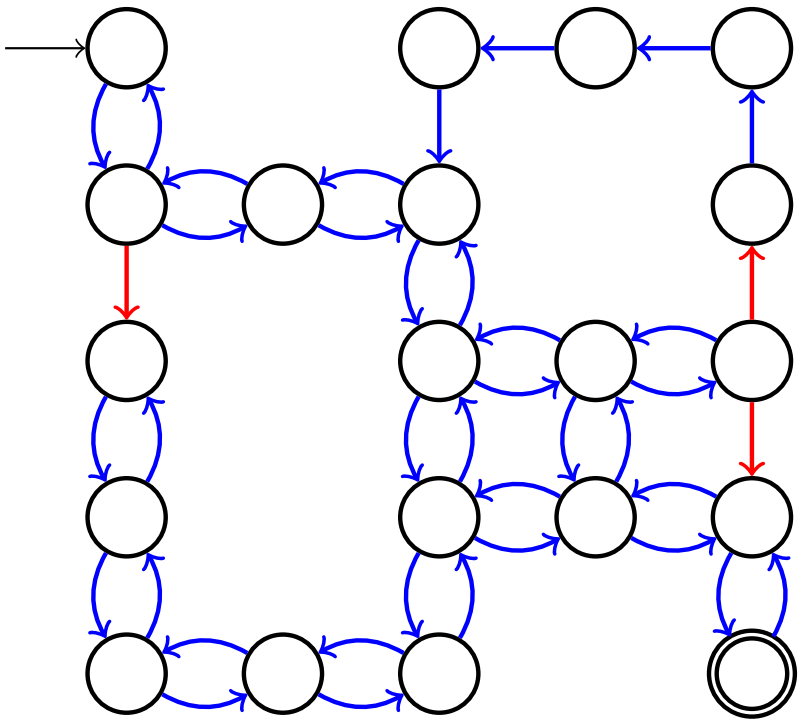

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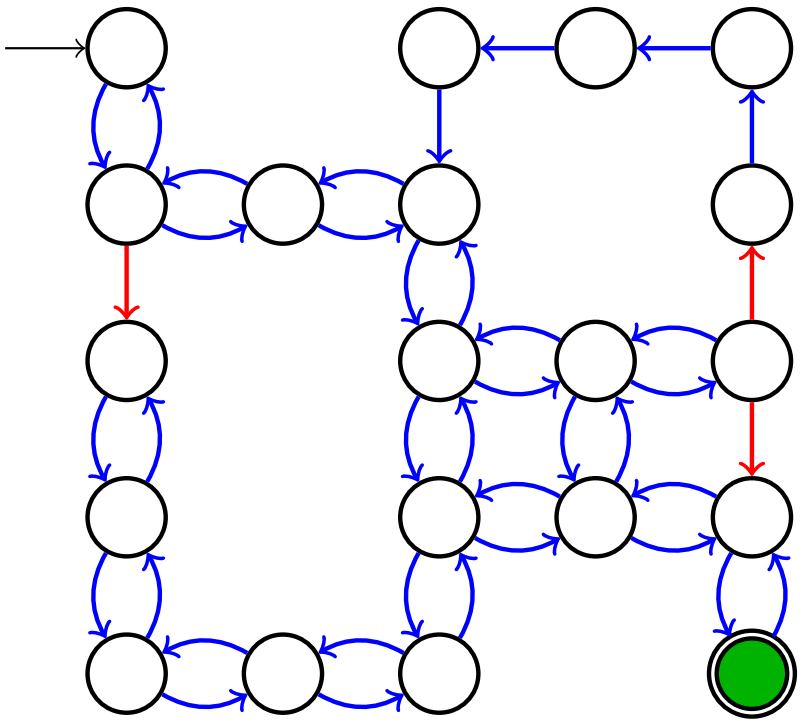

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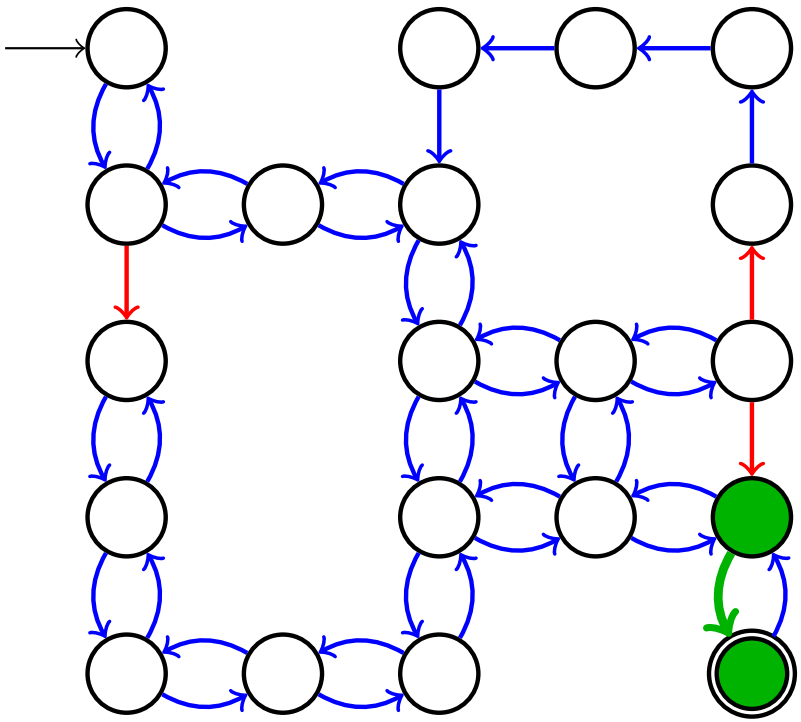

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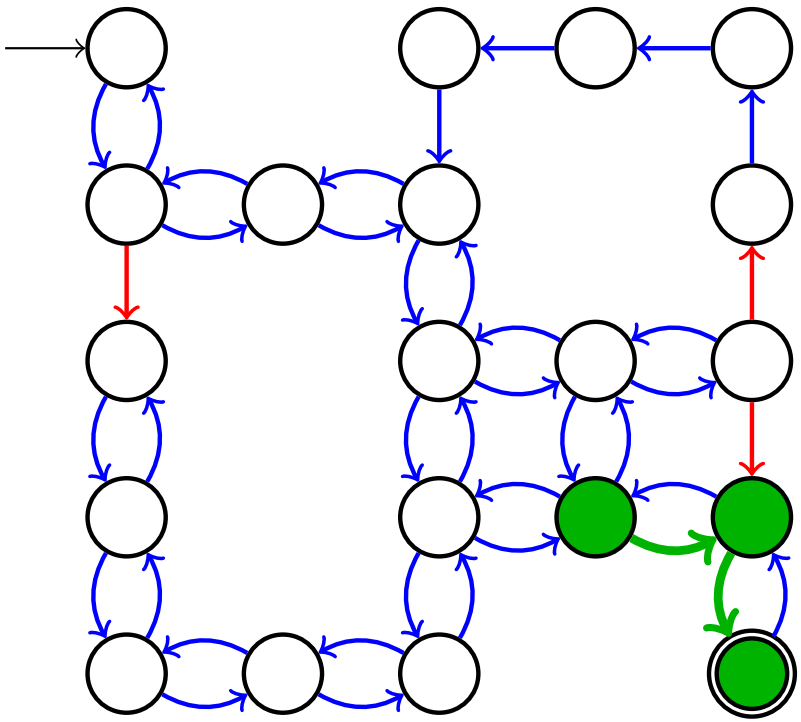

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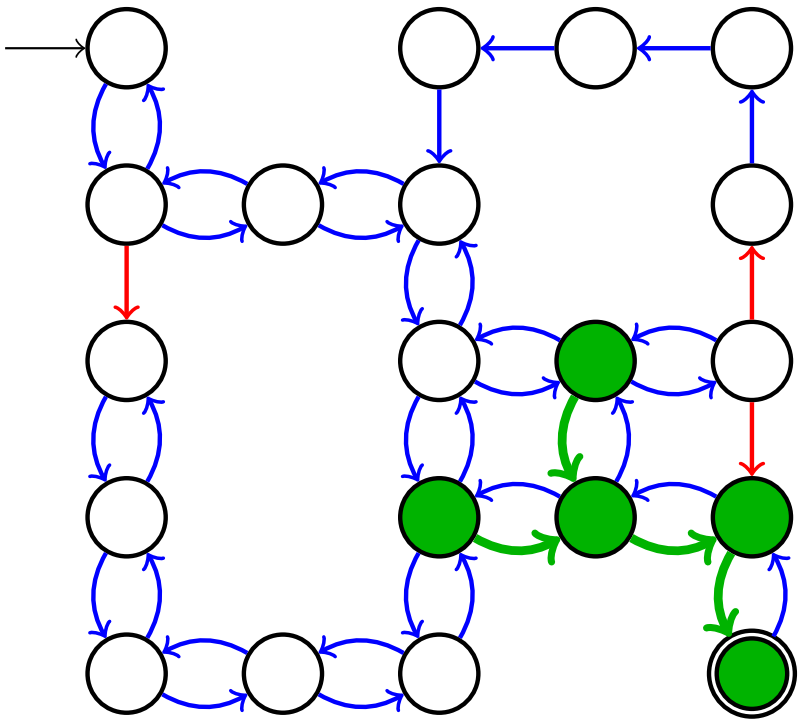

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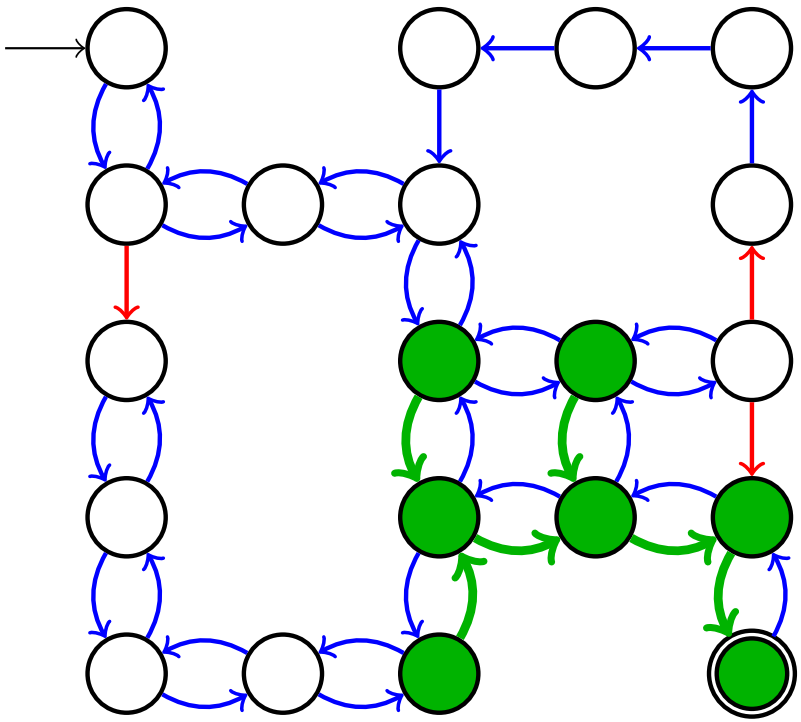

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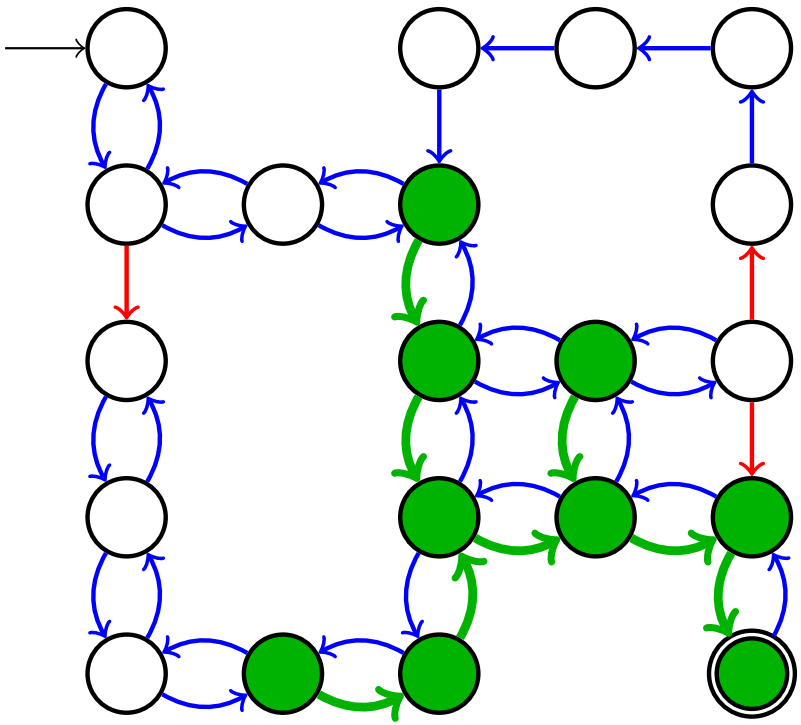

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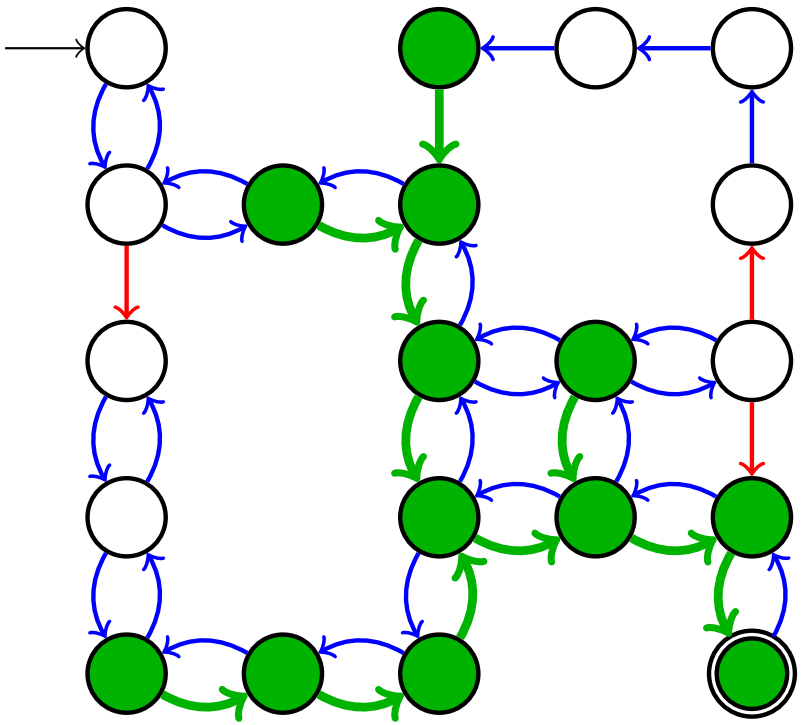

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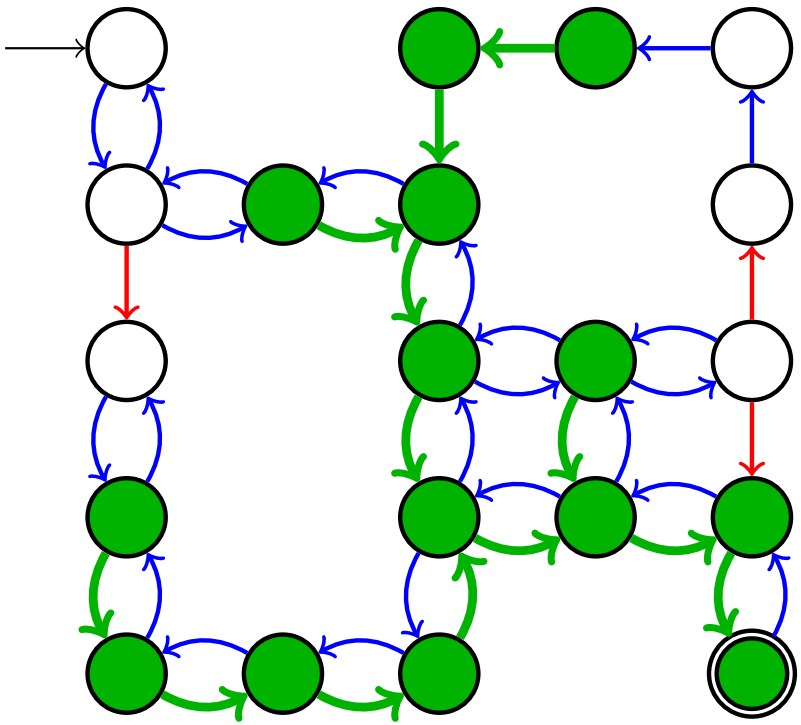

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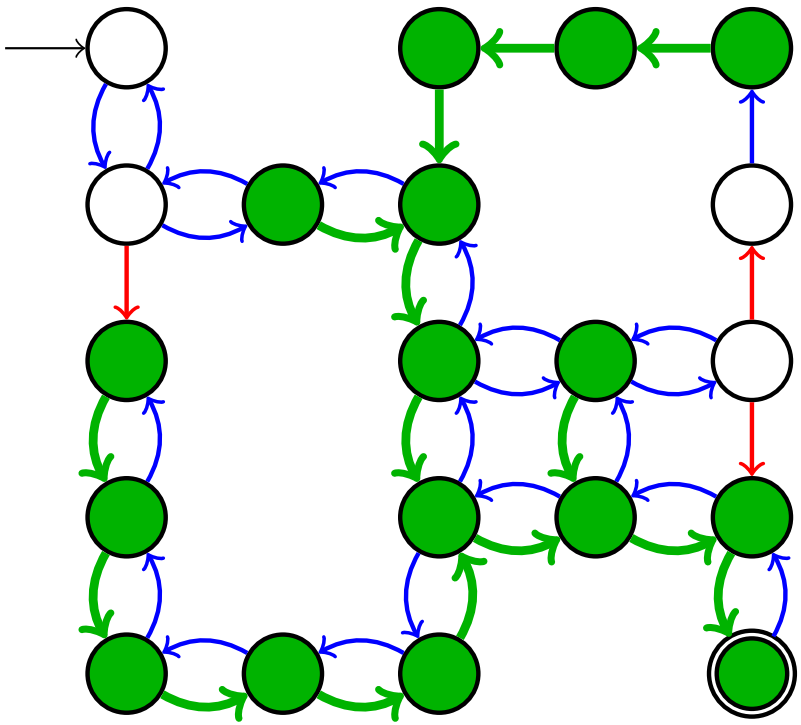

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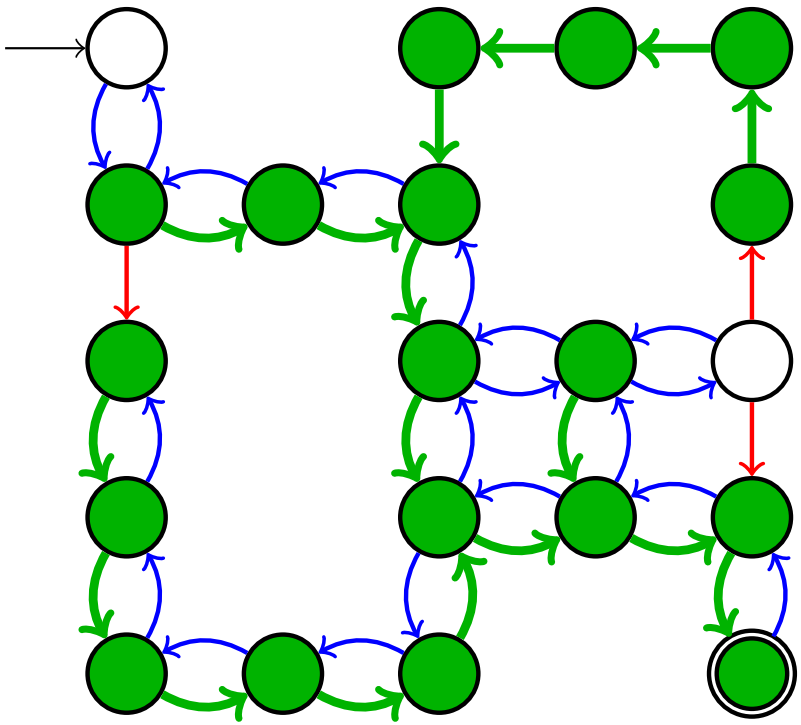

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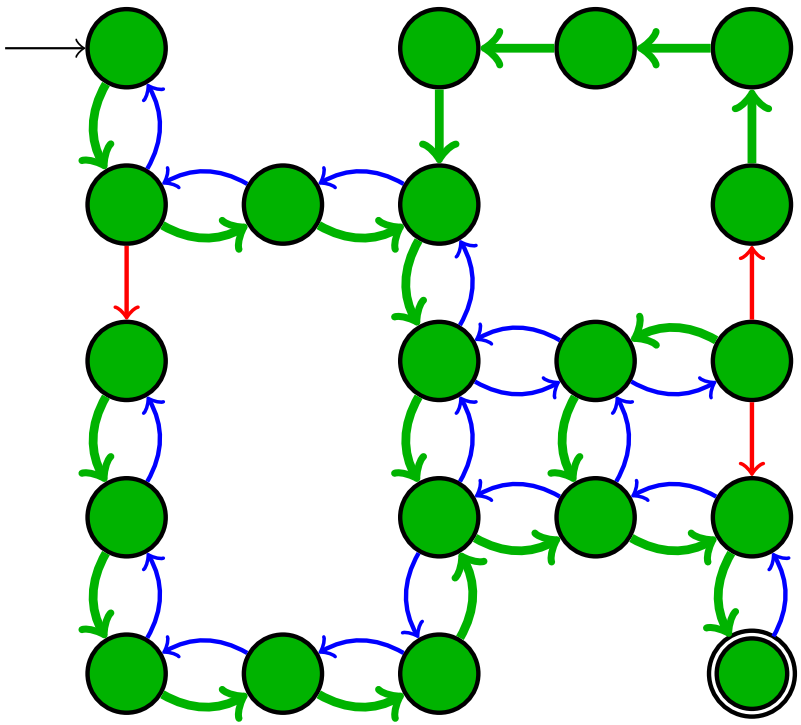

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  ...and add a controllable copy (because the environment could just refuse to play.)

Two-player safety game

- The reachability game is : there exists a strategy for the controller such that all maximal plays contain a goal state ;

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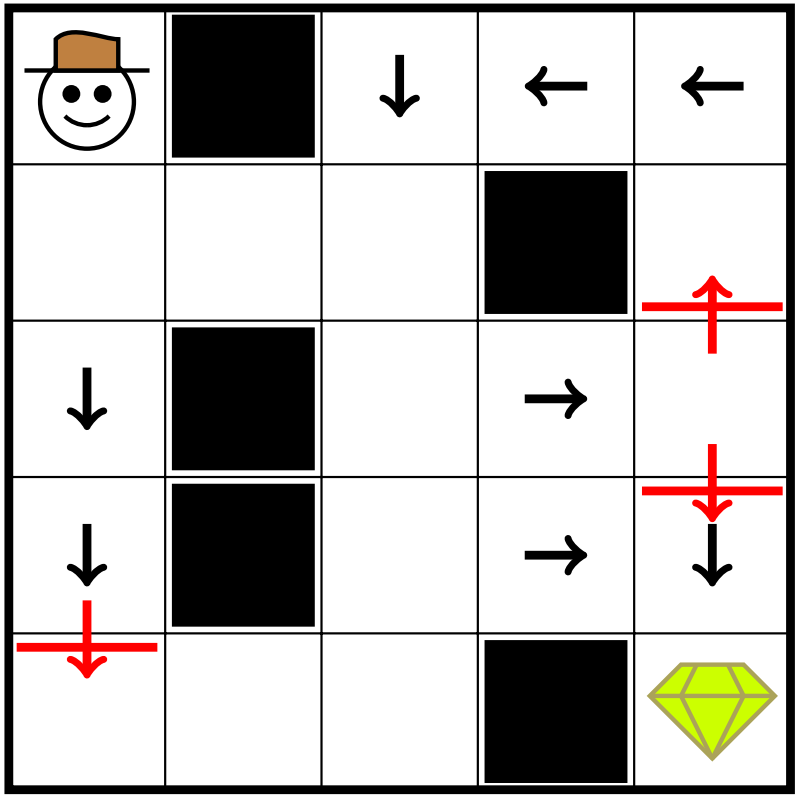

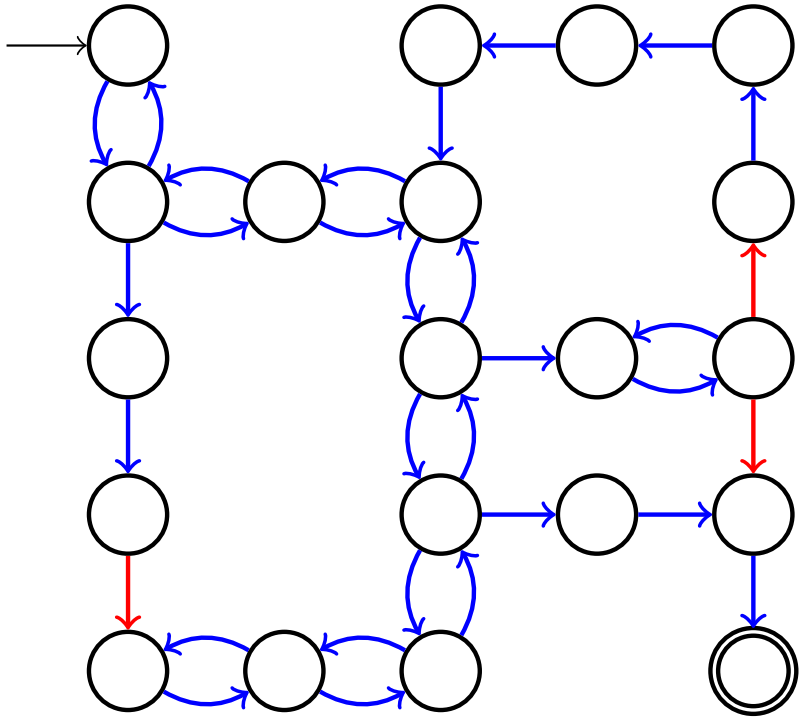

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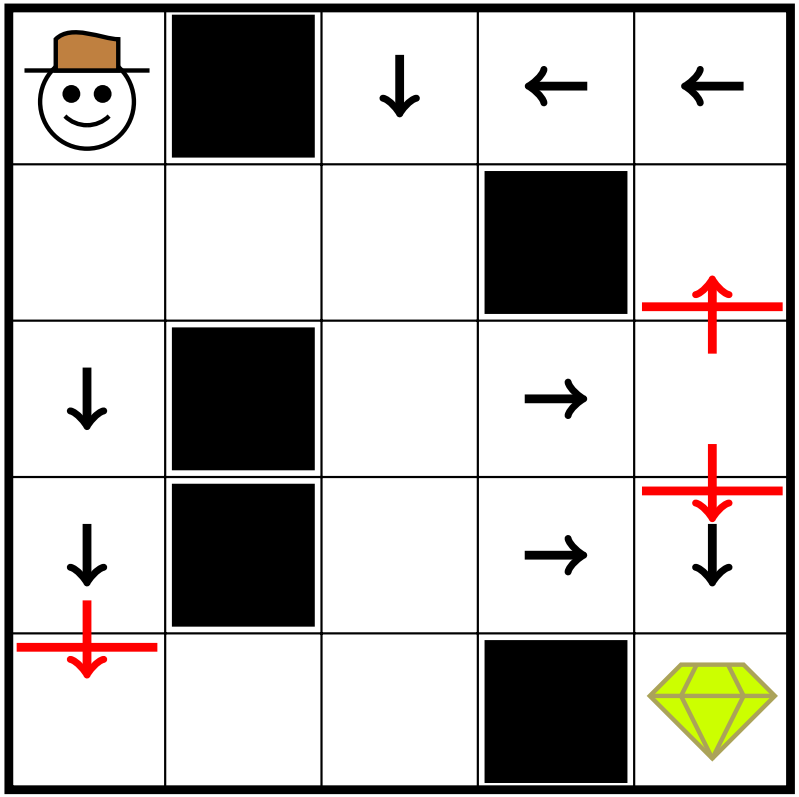

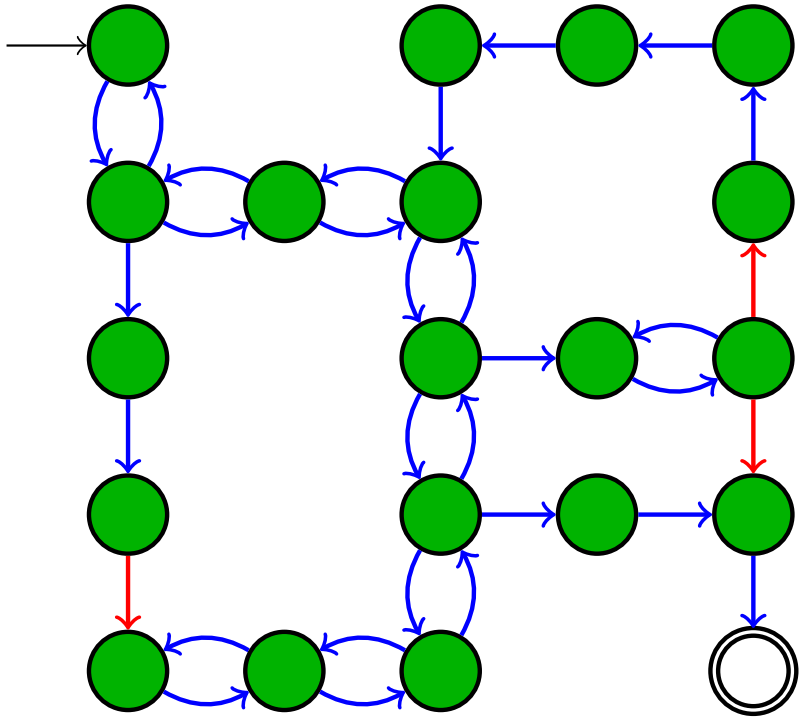

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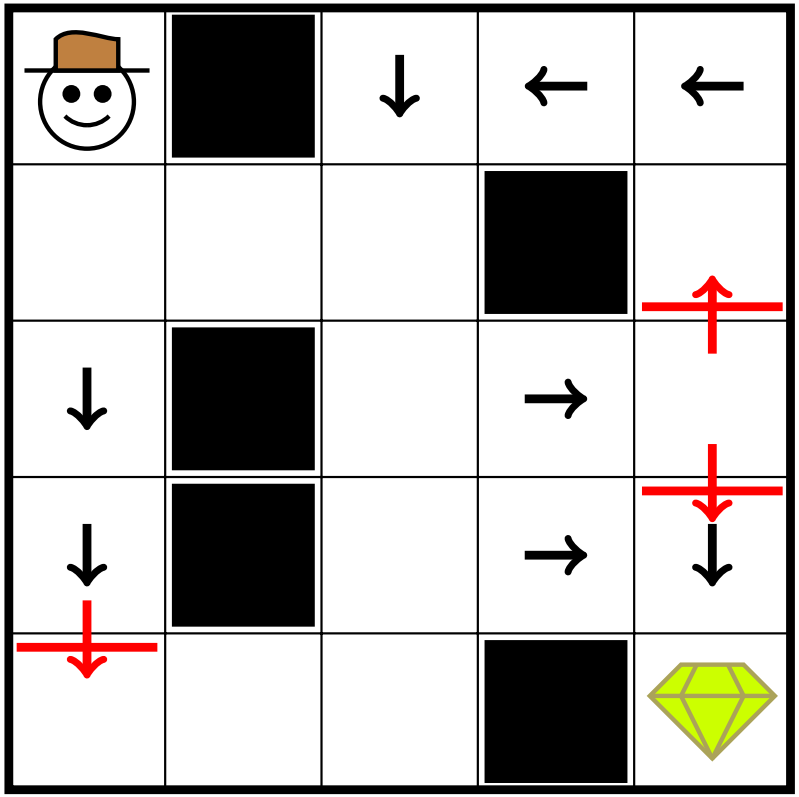

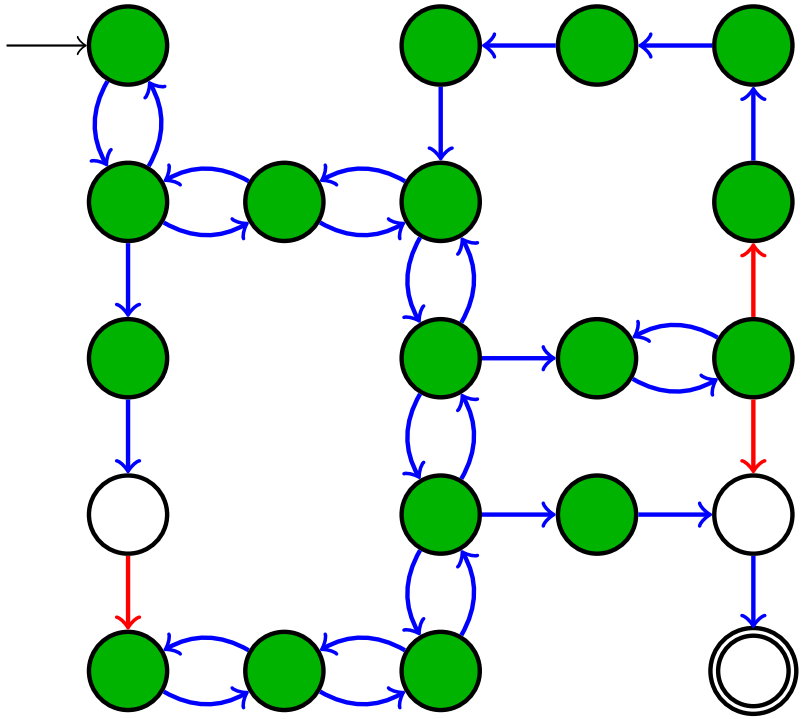

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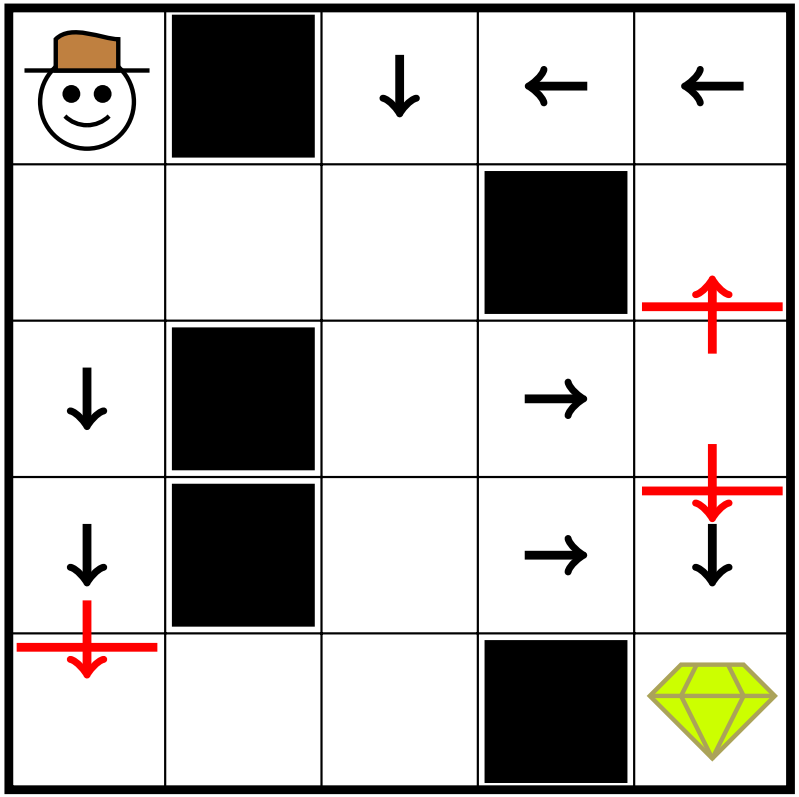

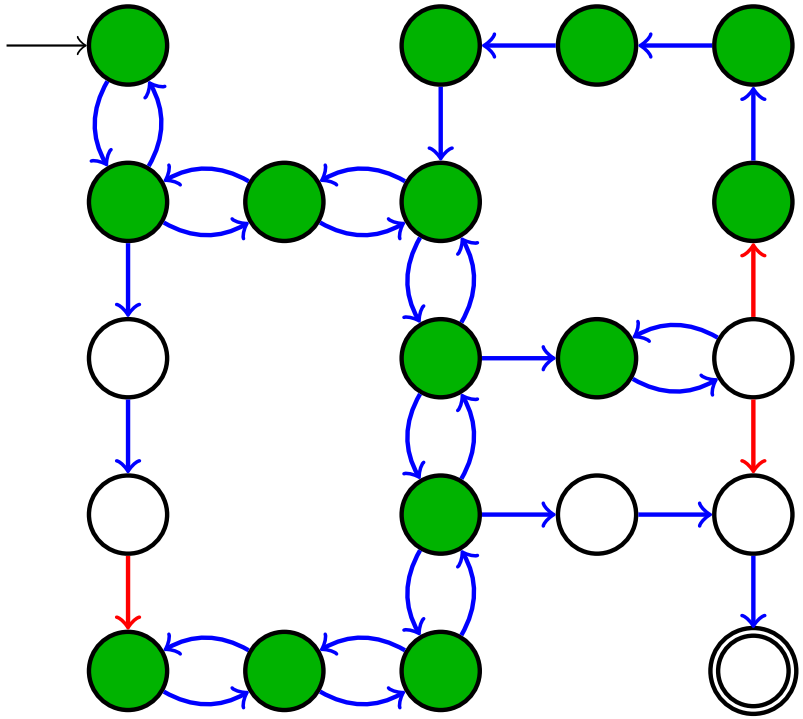

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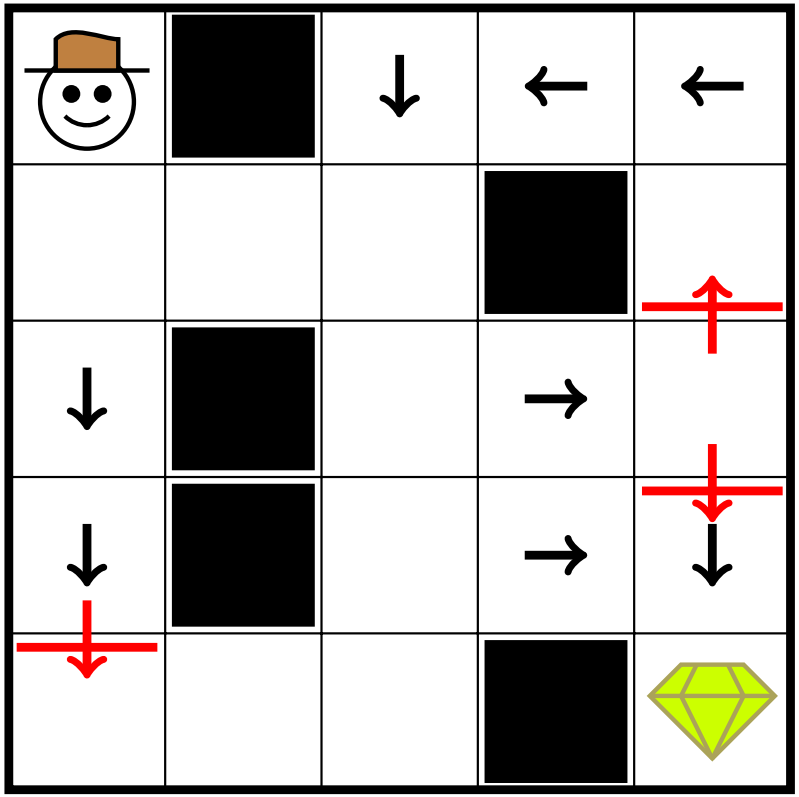

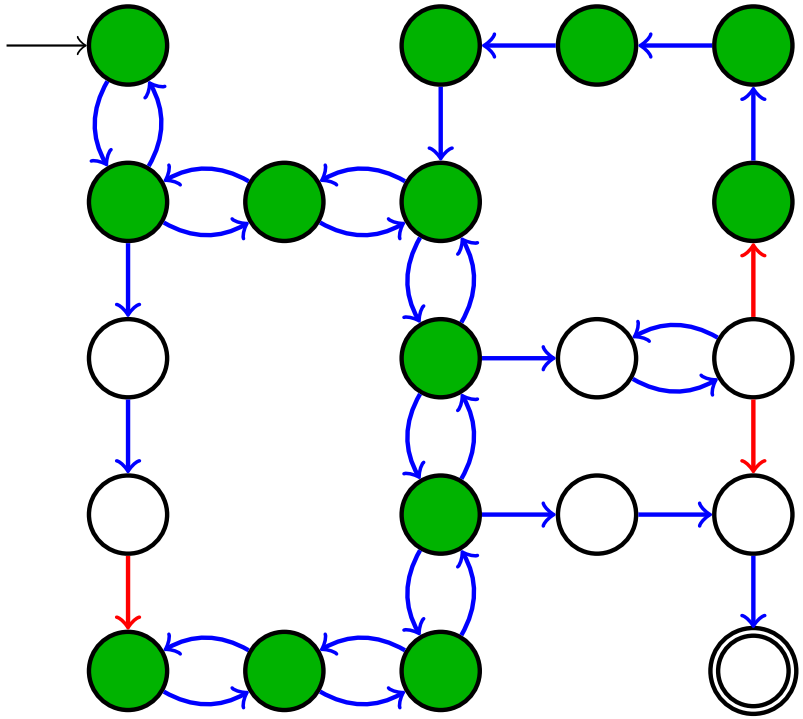

Two-player safety game

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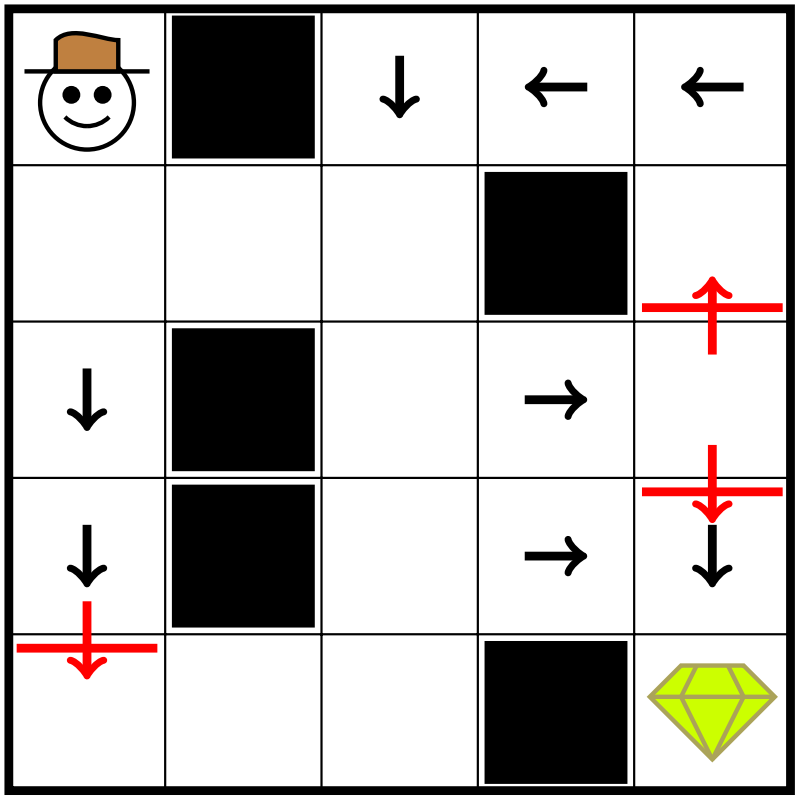
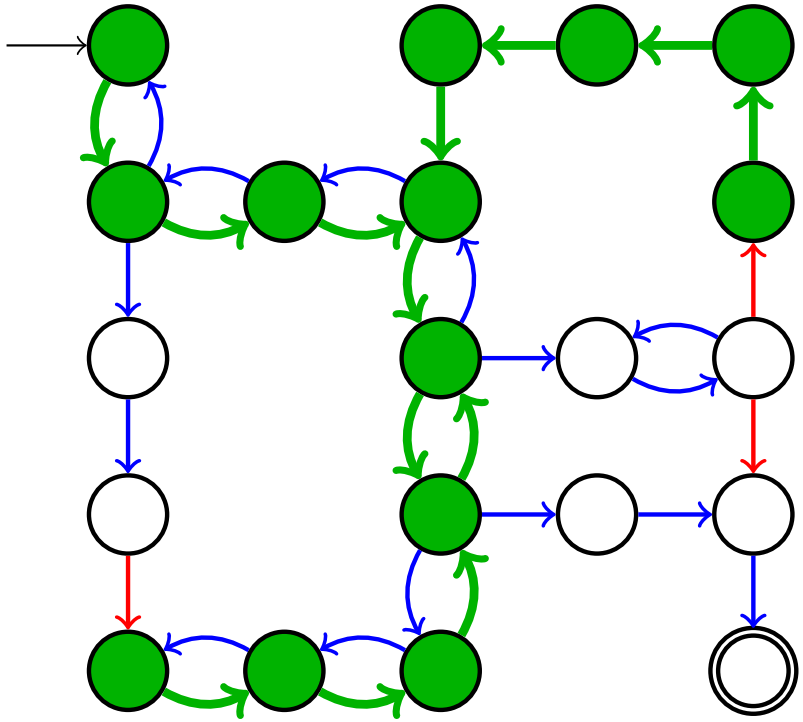
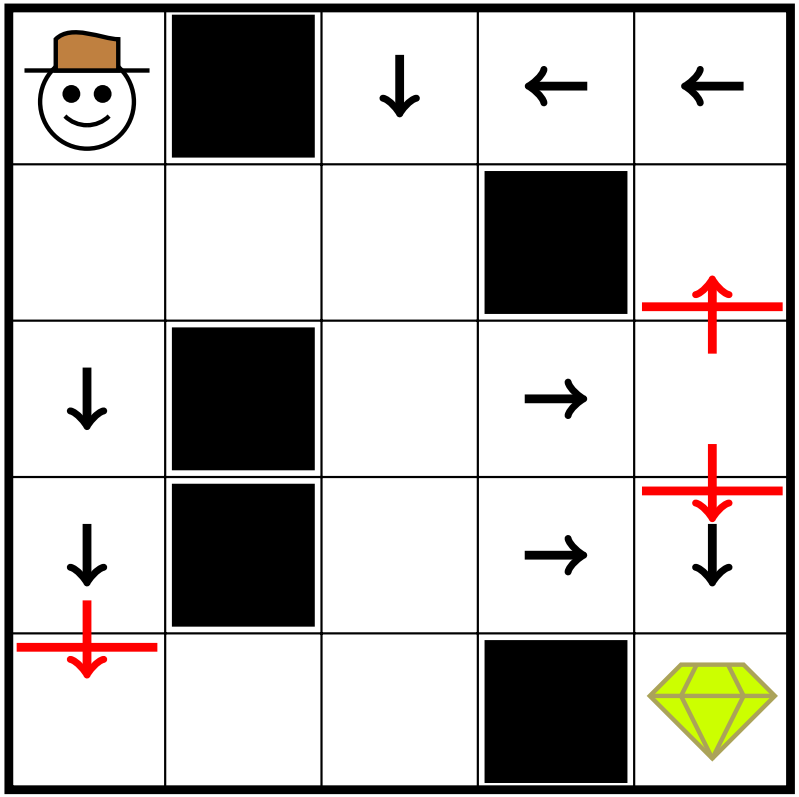


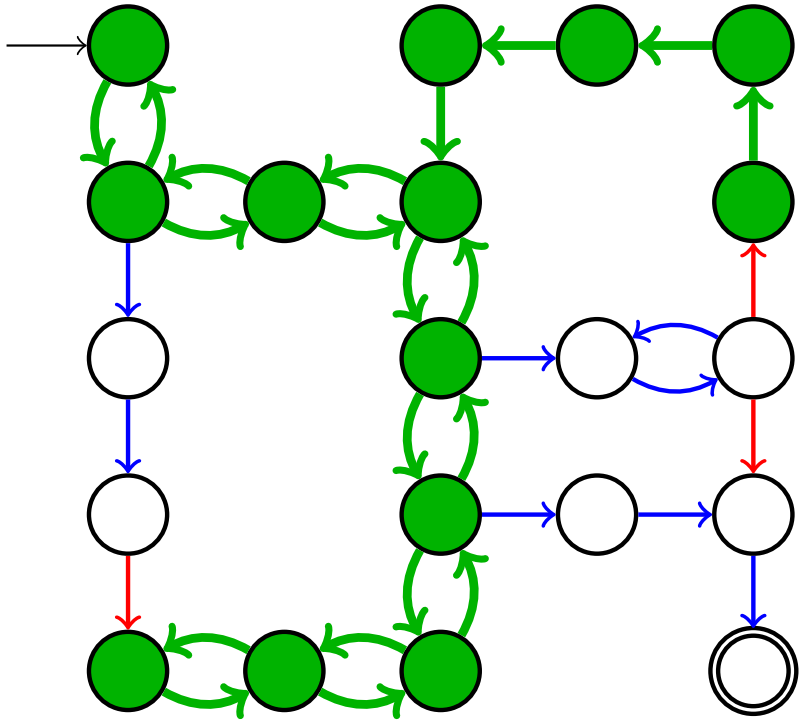


In winning states, a winning strategy is any controllable action to a winning state.

Two-player safety game

- Does  have a strategy to avoid  (it's a trap !) and being unable to move ?

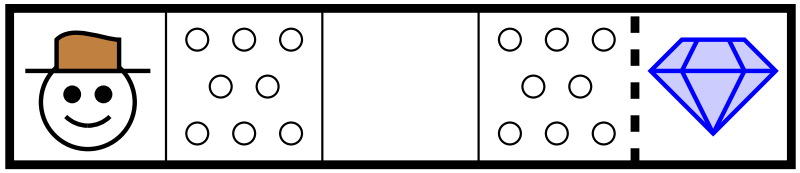




Any strategy that always goes to winning states is winning, memoryless or not ! We can define a most permissive strategy (which is not a strategy...)

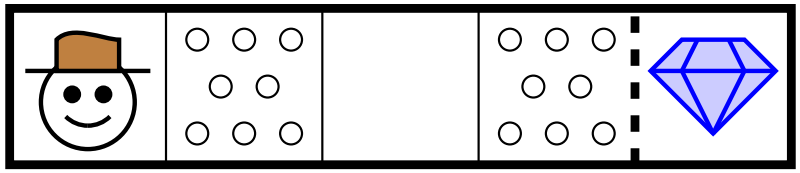
Two-player Timed Games

Timed games



1. The door to the  closes after 10 t.u. ;
2. Spikes () thrust upward with a period between 3 and 4 t.u. ;
3.  needs at least 1 t.u. to move to the next tile.

Timed games



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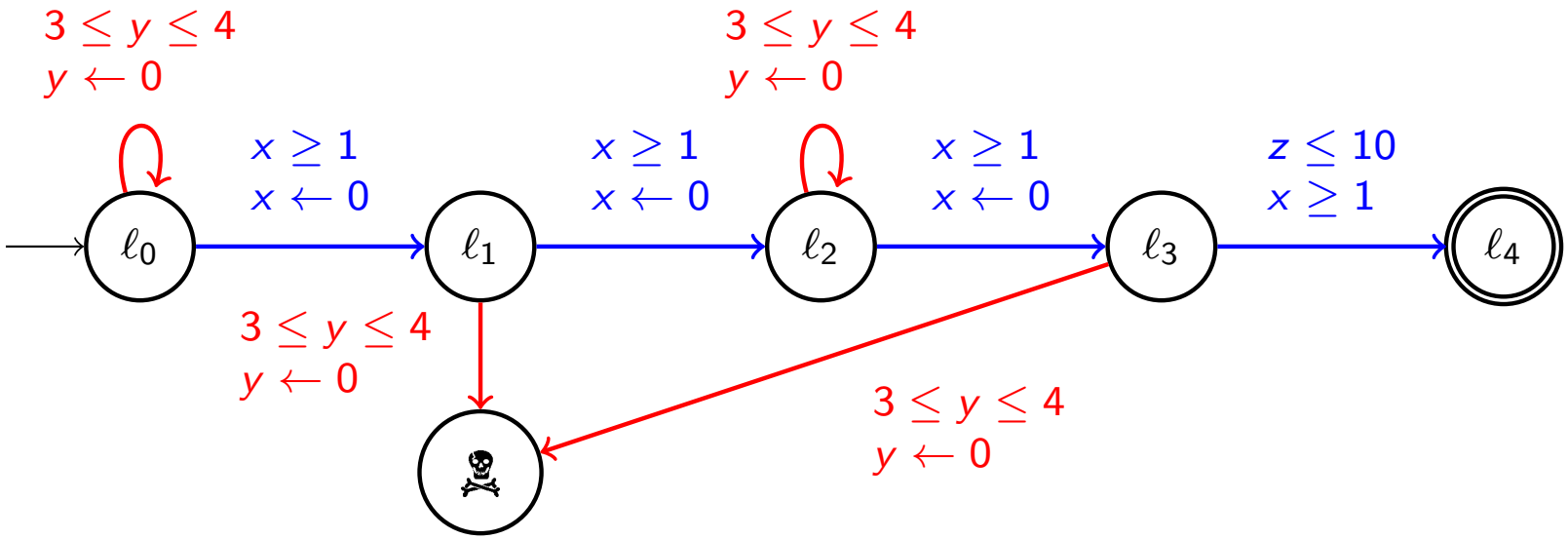
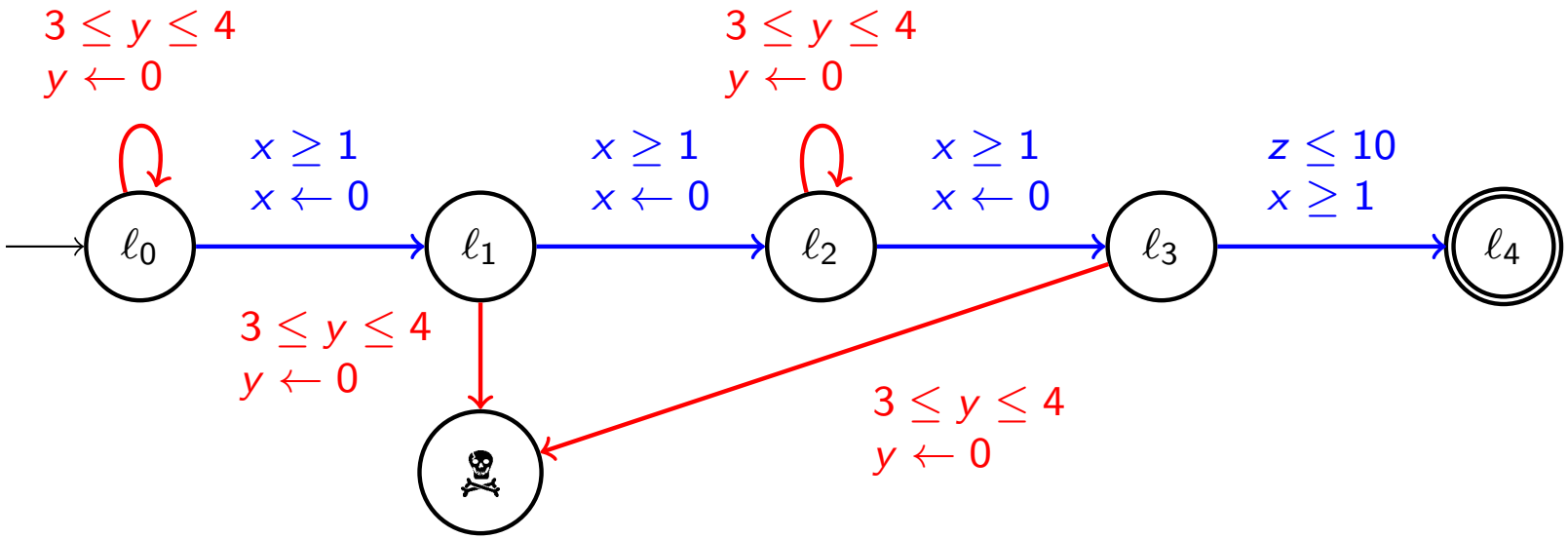


Strategies in timed games



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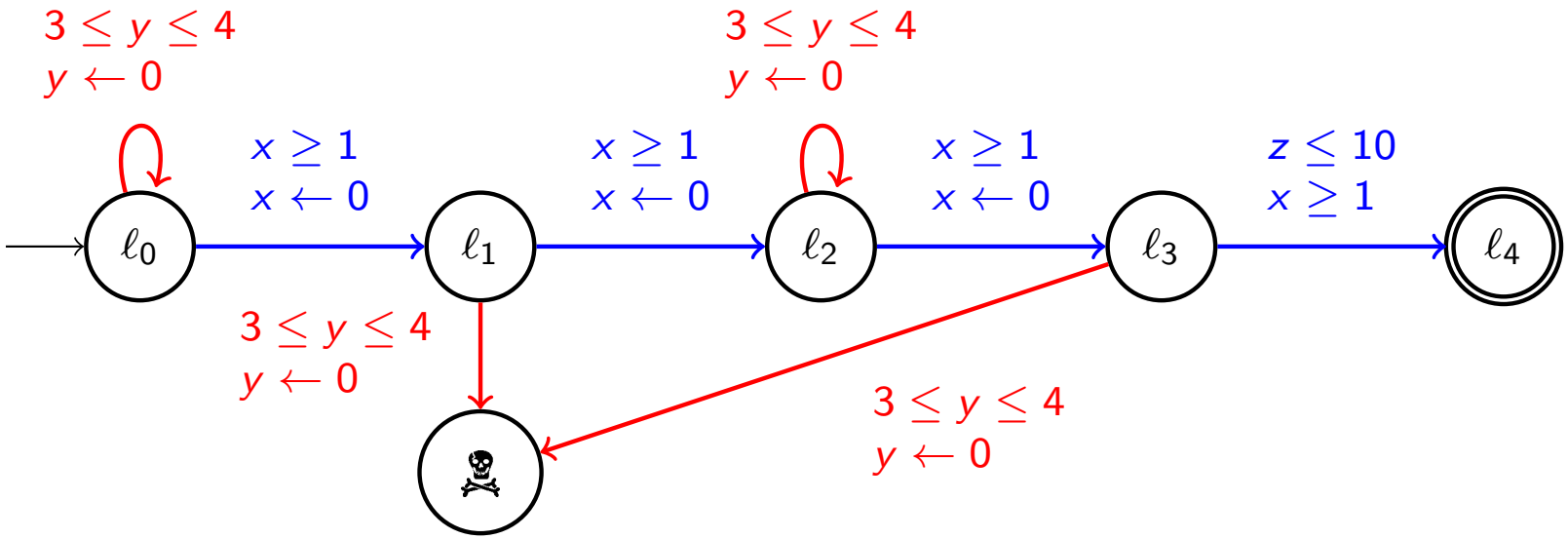

- States : (ℓ, x, y, z) ;

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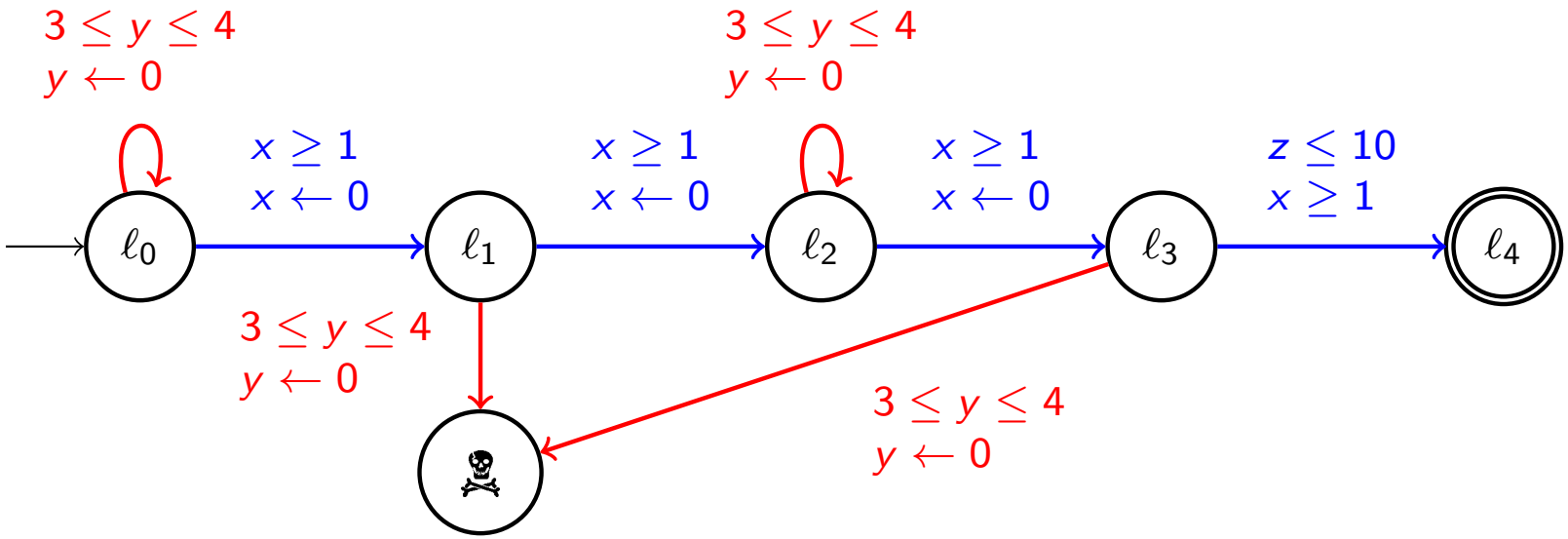

- States : (ℓ, x, y, z) ;
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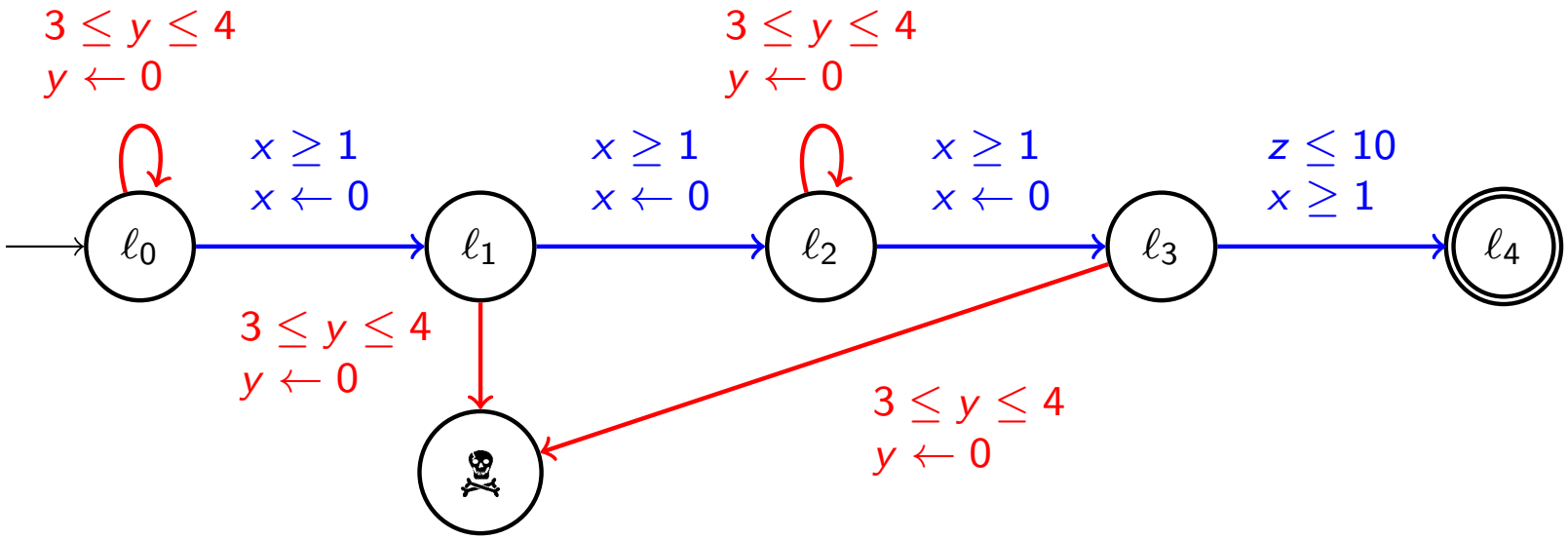

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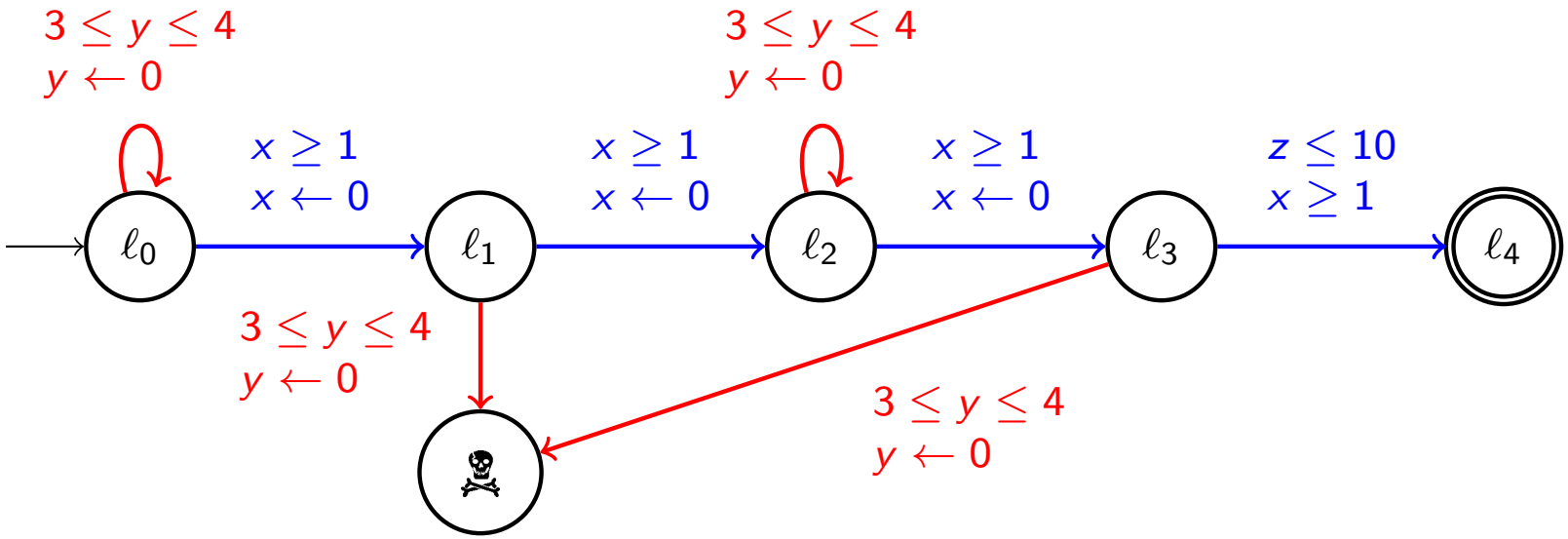

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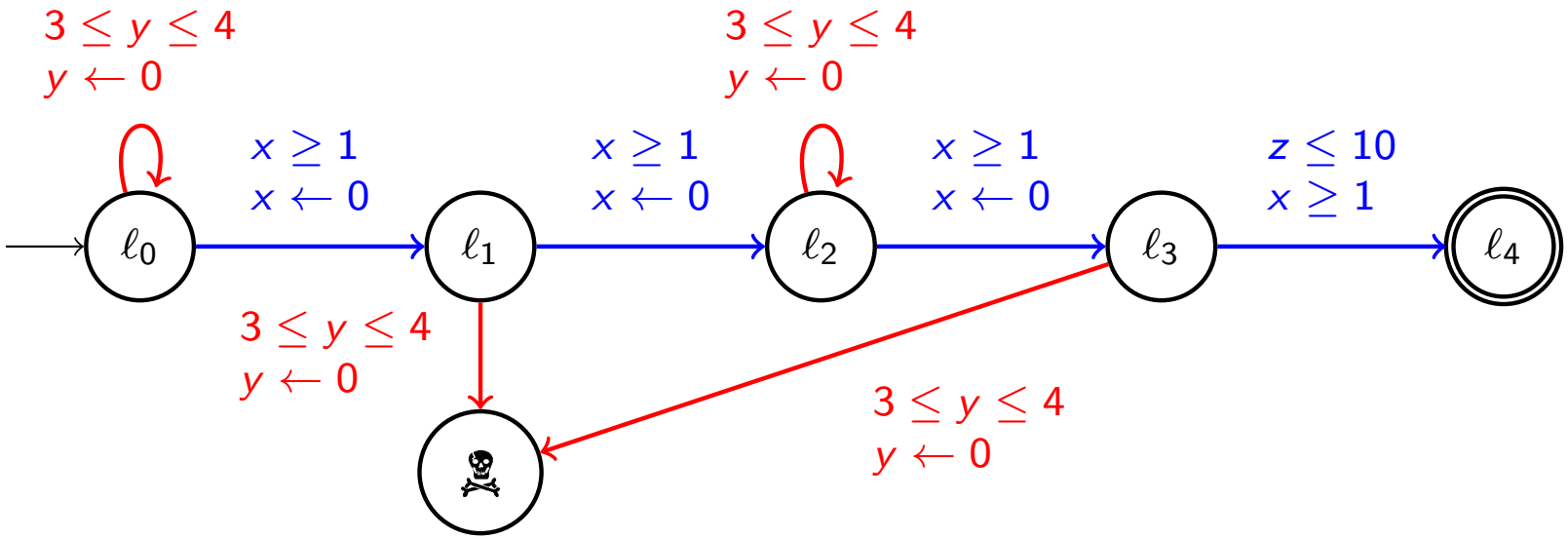

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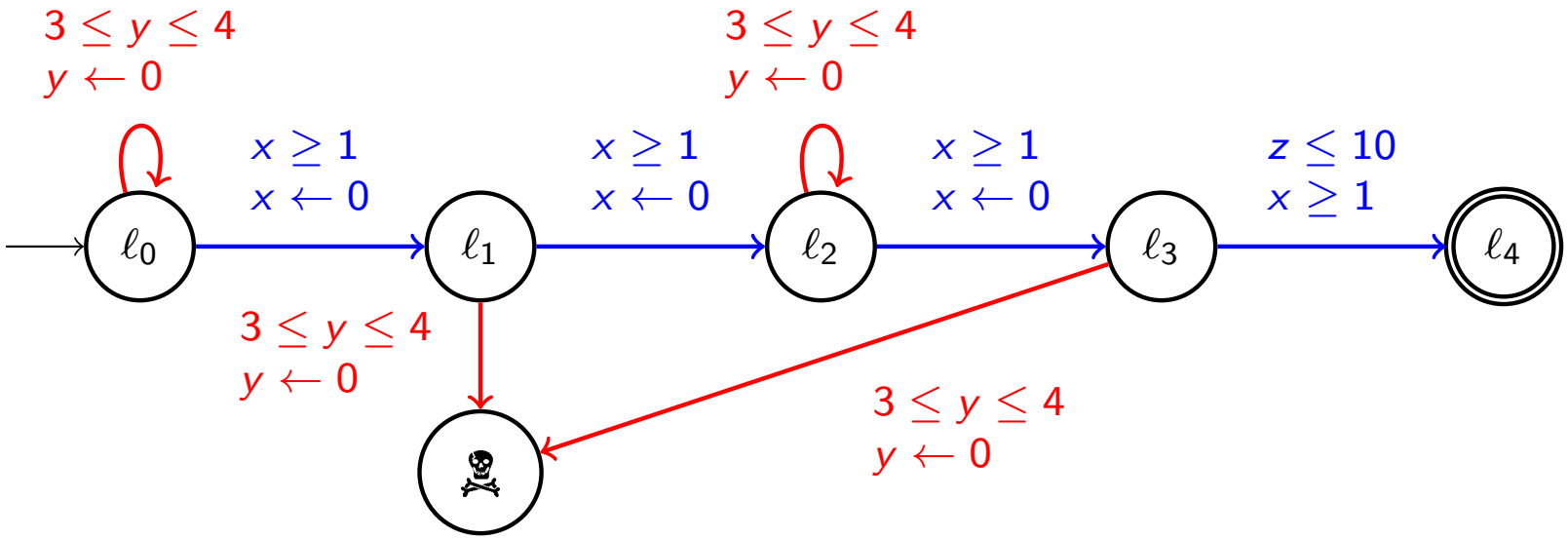

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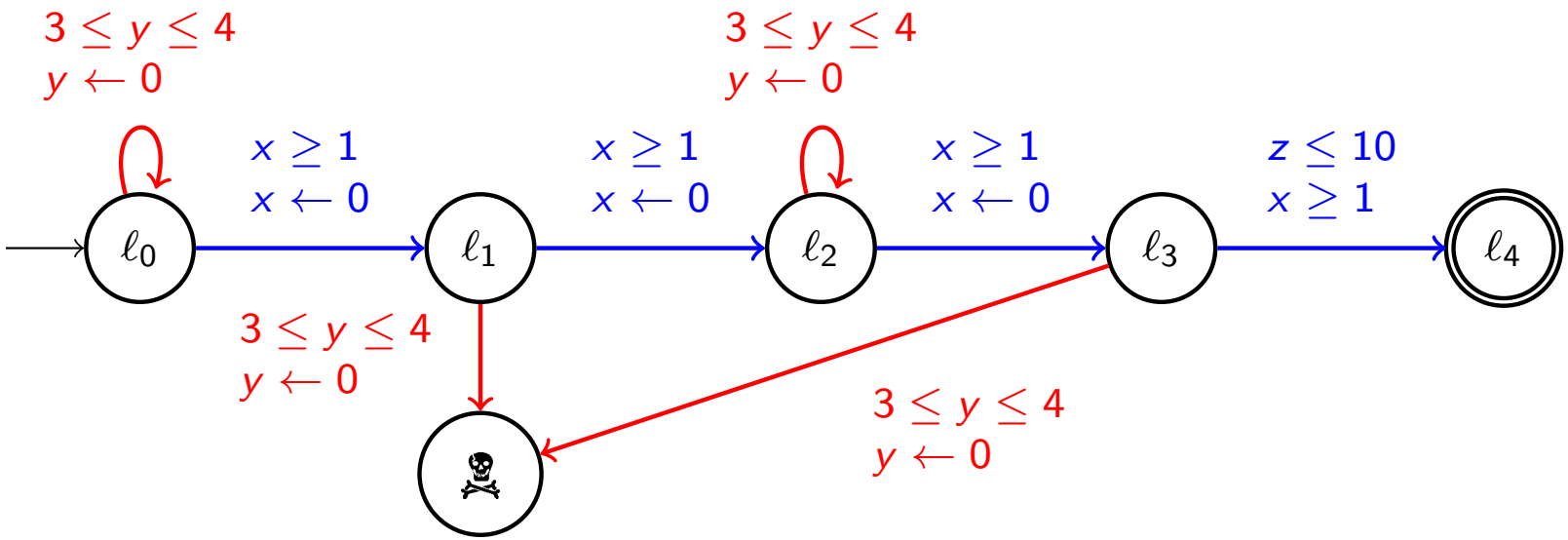
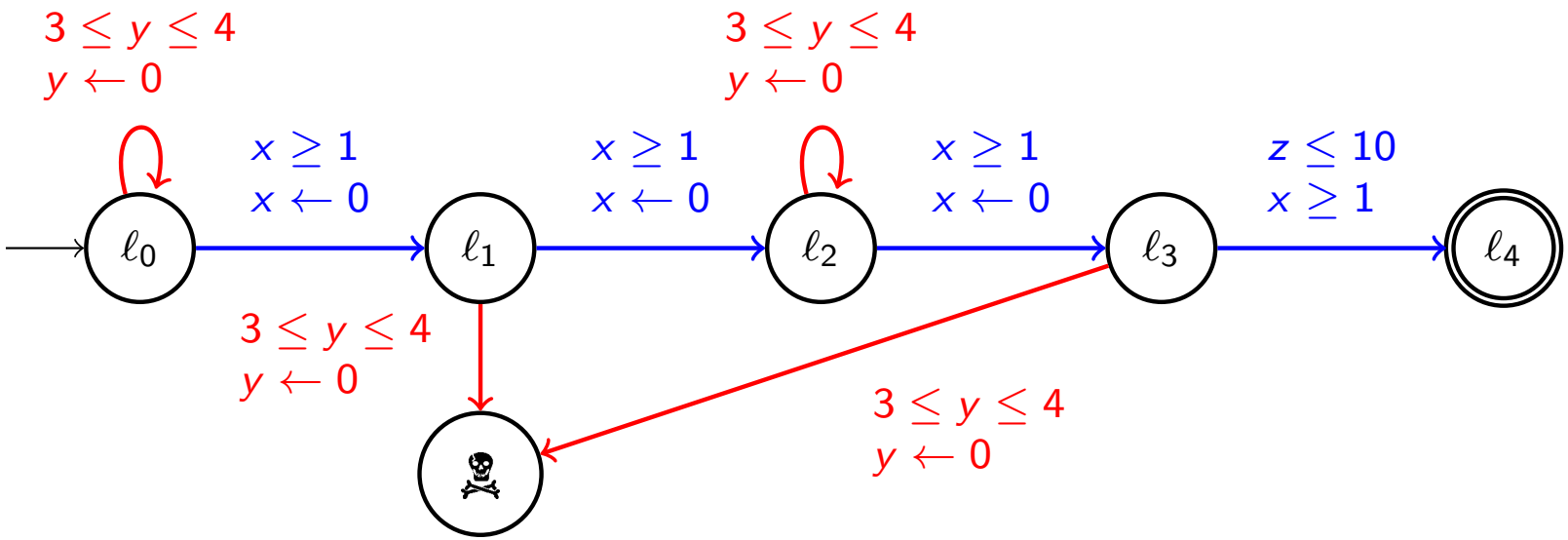
- If both play an action, one is chosen non-deterministically (as before) ;
- We assume strategies are constant on regions.

Plays in timed games



- We write :
  $(\ell_0, 0, 0, 0) \xrightarrow{@3.2} (\ell_1, 0, 3.2, 3.2)$ for $(\ell_0, 0, 0, 0) \xrightarrow{3.2} (\ell_0, 3.2, 3.2, 3.2) \rightarrow (\ell_1, 0, 3.2, 3.2)$
- Example plays :

Plays in timed games



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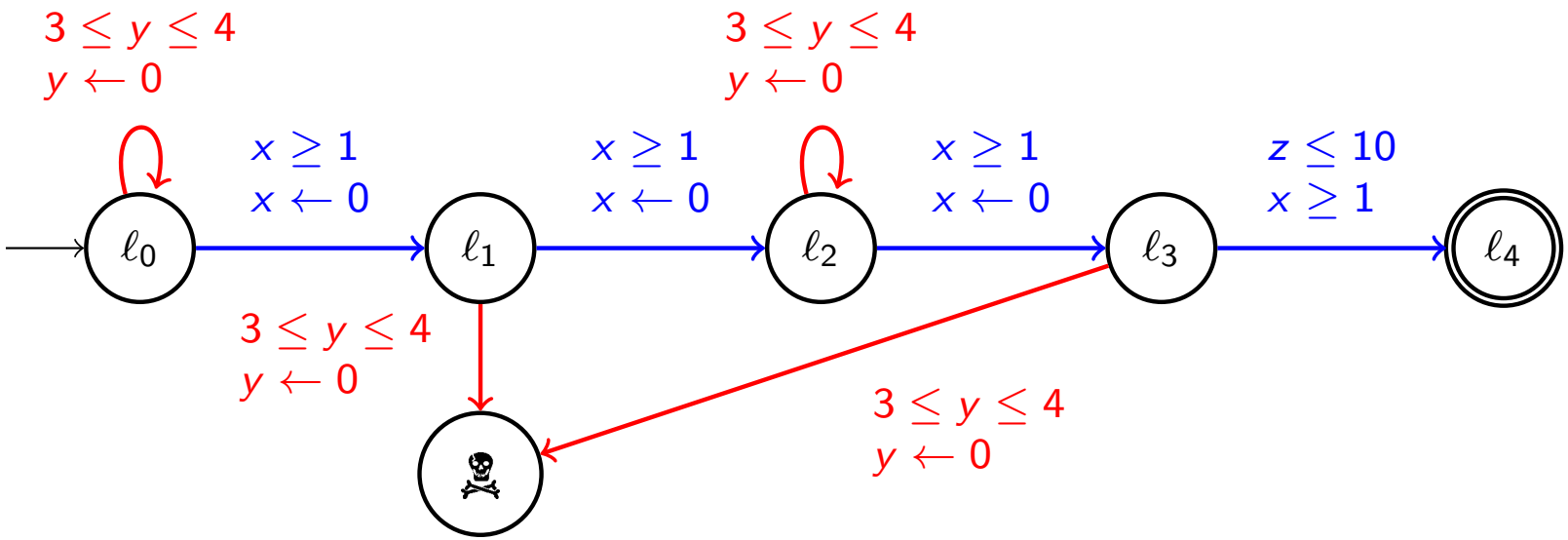
$$(\ell_0, 0, 0, 0) \xrightarrow{@3.2} (\ell_1, 0, 3.2, 3.2) \xrightarrow{@1} (\ell_2, 0, 4.2, 4.2) \xrightarrow{@1.3} (\ell_3, 0, 5.5, 5.5) \xrightarrow{@1.4} (\ell_4, 1.4, 6.9, 6.9)$$

Plays in timed games



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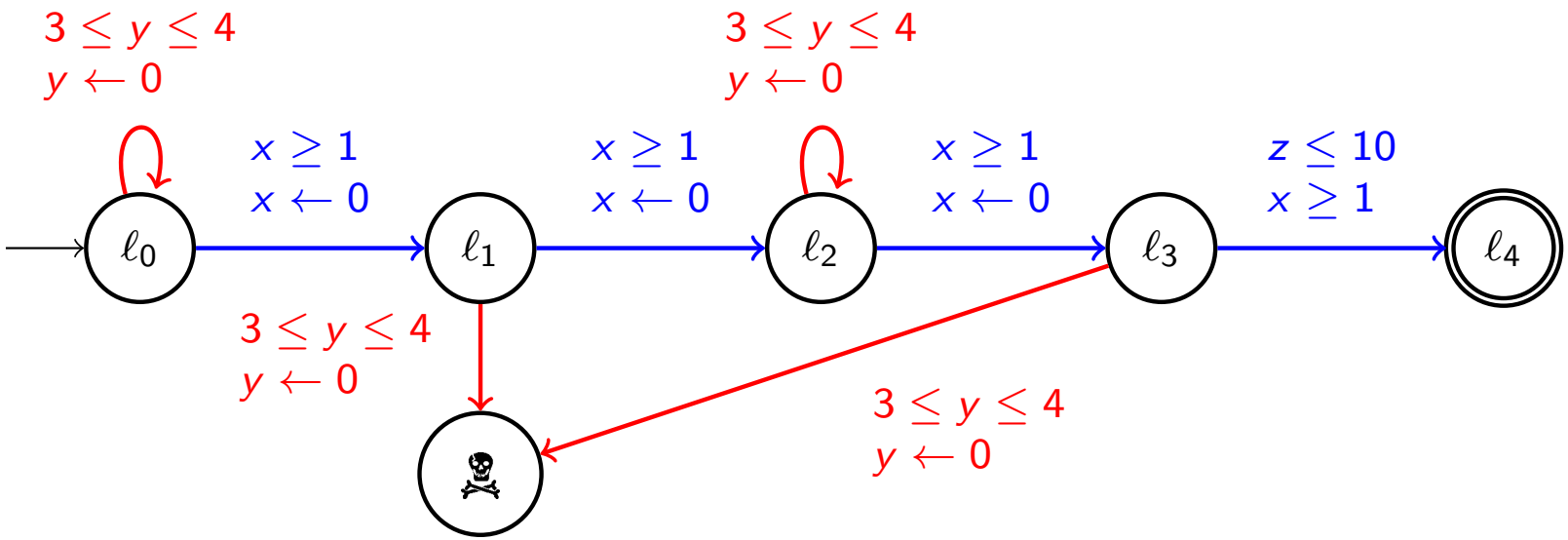
$(\ell_0, 0, 0, 0) \xrightarrow{@3.2} (\ell_1, 0, 3.2, 3.2) \xrightarrow{@0.5} (☠, 0.5, 0, 3.7) \xrightarrow{10.7} (☠, 11.2, 10.7, 14.4)$

Plays in timed games



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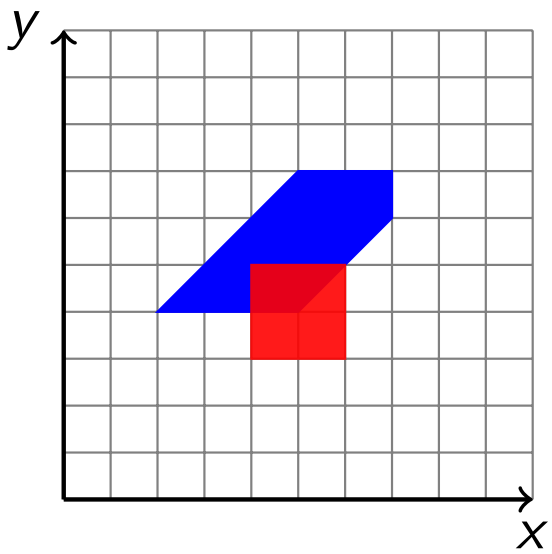
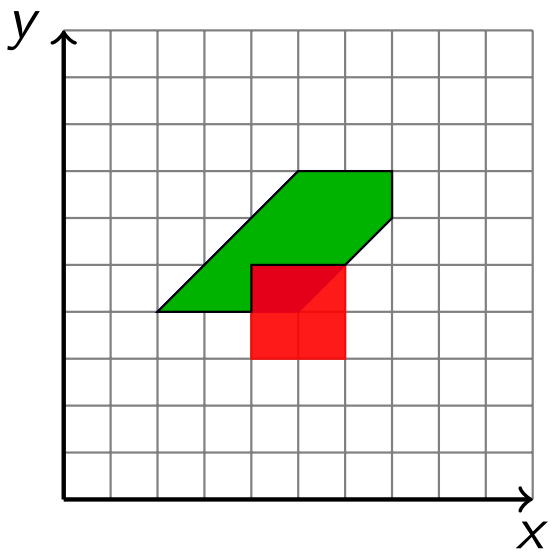
$(\ell_0, 0, 0, 0) \xrightarrow{@1.9} (\ell_1, 0, 1.9, 1.9) \xrightarrow{@1} (\ell_2, 0, 2.9, 2.9) \xrightarrow{@0.1} (\ell_2, 0.1, 0, 3) \xrightarrow{@1} (\ell_3, 0, 1, 4) \xrightarrow{@1} (\ell_4, 1, 2, 5)$

Computing winning states

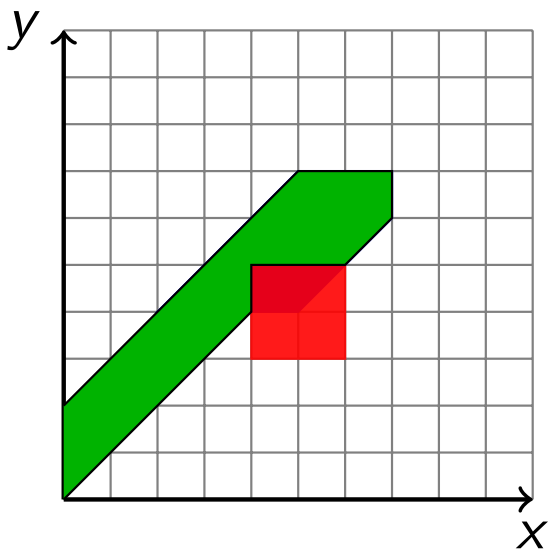


Computing winning states



- As before, if W are winning states, states in $\mathrm{Pre}_c(W) \setminus \mathrm{Pre}_u(\overline{W})$ are winning too.

Computing winning states



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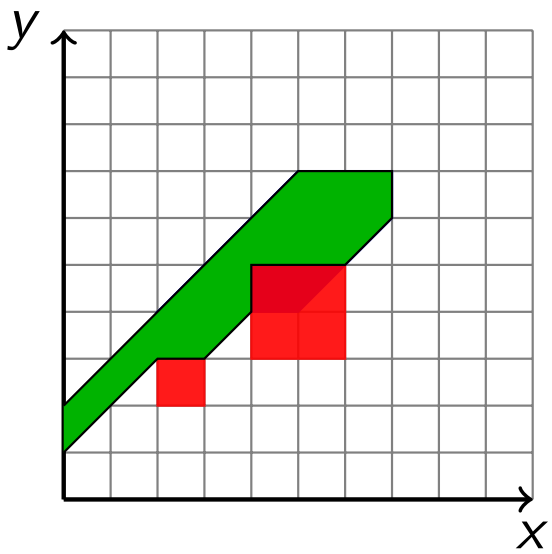
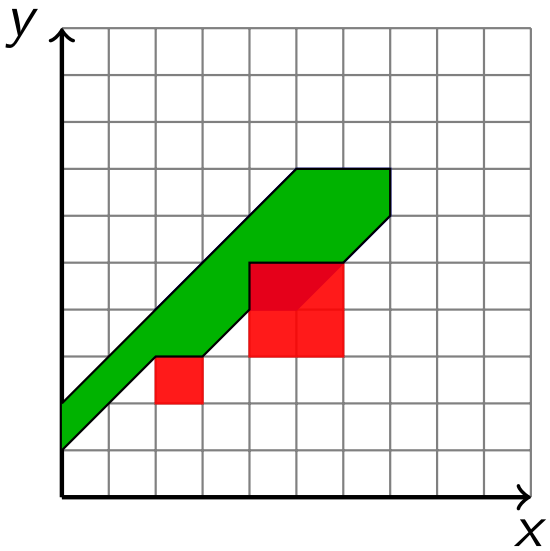
- But also states that can reach those by delay ;

Computing winning states



- As before, if W are winning states, states in $\mathrm{Pre}_c(W) \setminus \mathrm{Pre}_u(\overline{W})$ are winning too.
- But also states that can reach those by delay ;
- Provided they avoid losing/uncertain states on the way.

Computing winning states



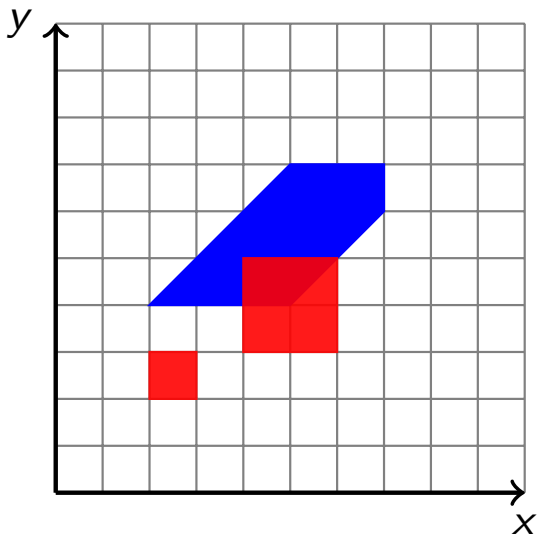
$$X_0 = G$$
$$X_{n+1} = X_n \cup \pi(X_n)$$

With :

$$\pi(X) = \mathrm{Pre_t}(\mathrm{Pre}_c(X), \mathrm{Pre}_u(\overline{X}))$$

$$\mathrm{Pre_t}(X, Y) = \{s \in S | \exists d \geq 0, s' \in X \text{ s. t. } s \xrightarrow{d} s' \text{ and } \forall 0 \leq d' \leq d, s \xrightarrow{d'} \notin Y\}$$

Computing a winning strategy



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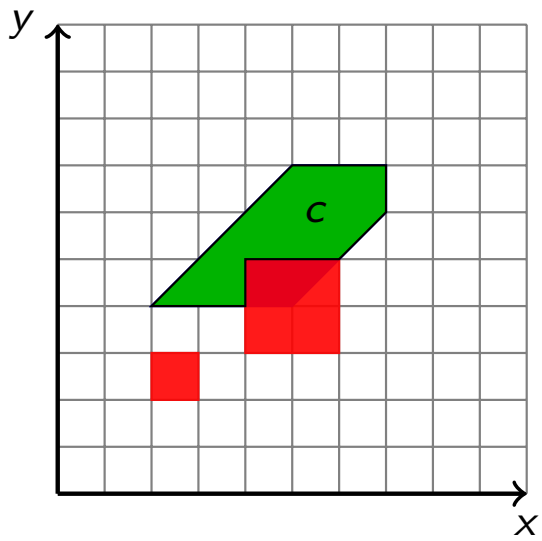

- For all states in $\mathrm{Pre}_c(W) \setminus \mathrm{Pre}_u(\overline{W})$, the strategy is the action leading to W.

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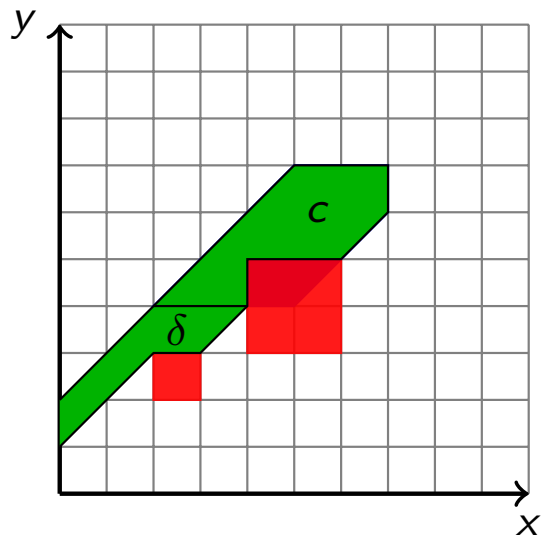
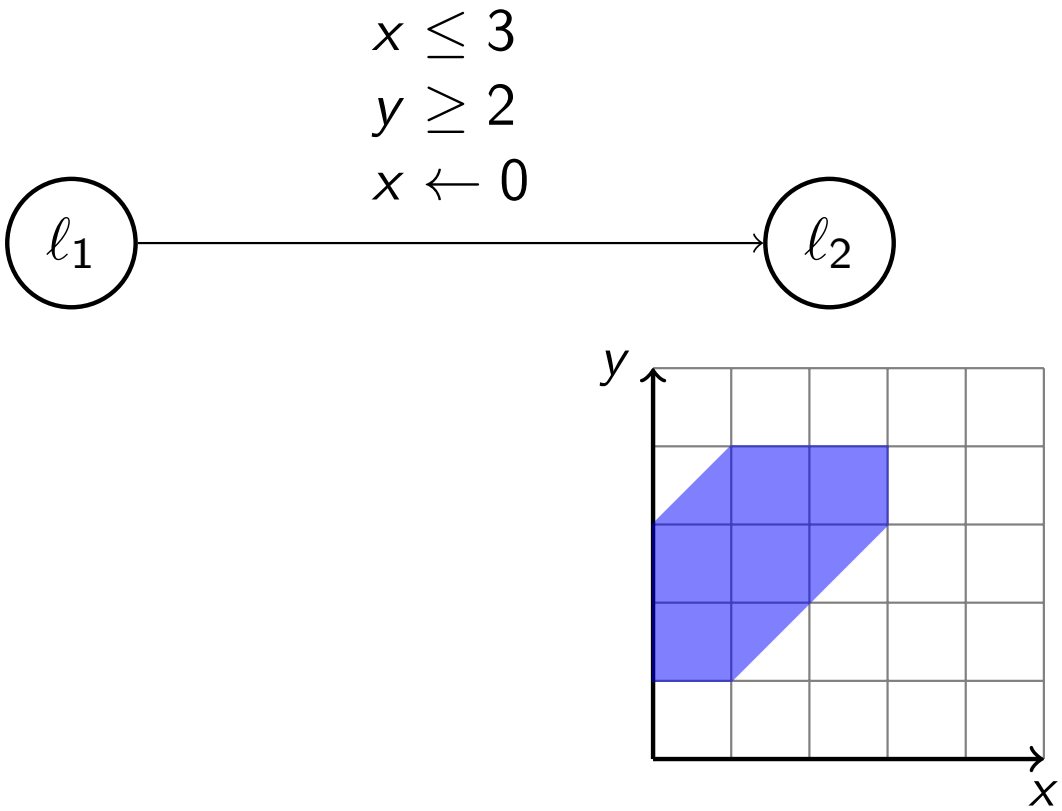


- For all states in $\mathrm{Pre}_c(W) \setminus \mathrm{Pre}_u(\overline{W})$, the strategy is the action leading to W.
- In all other winning states, the strategy is delay (δ).

Computing action predecessors (with zones)



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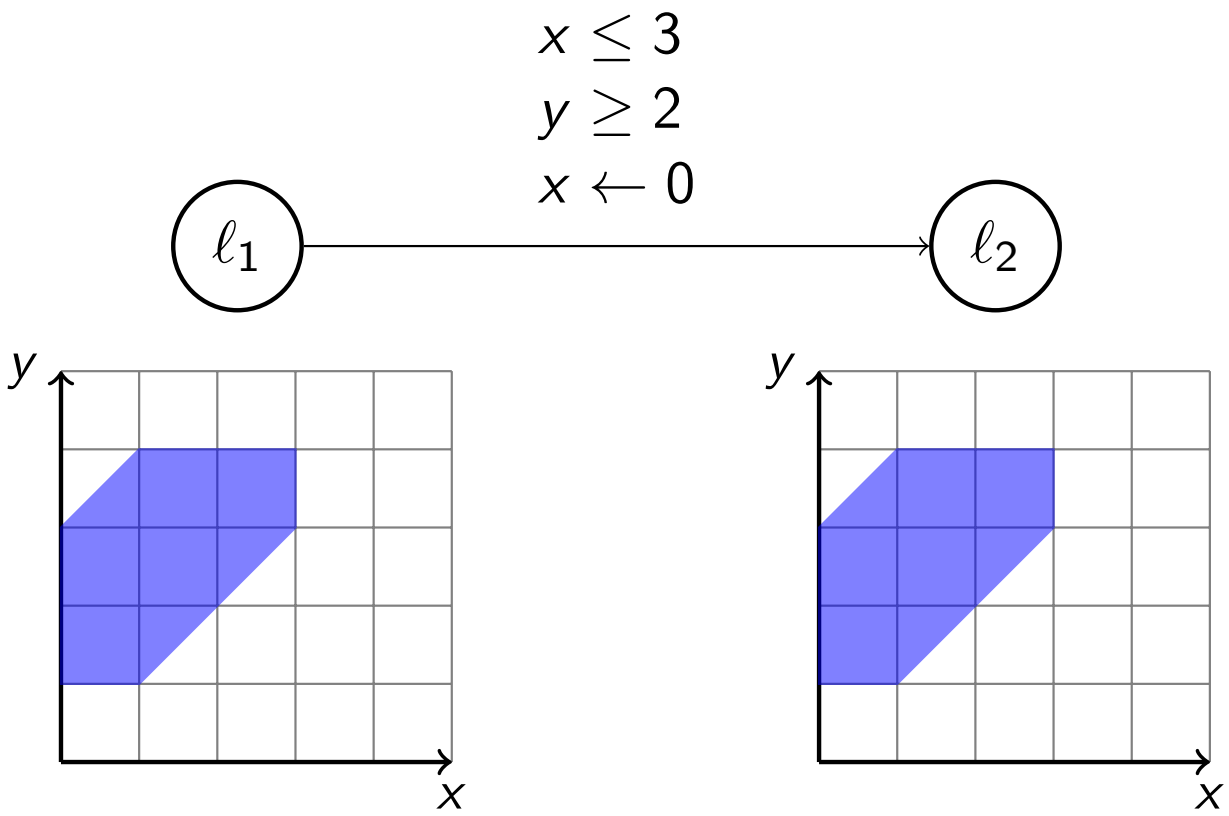

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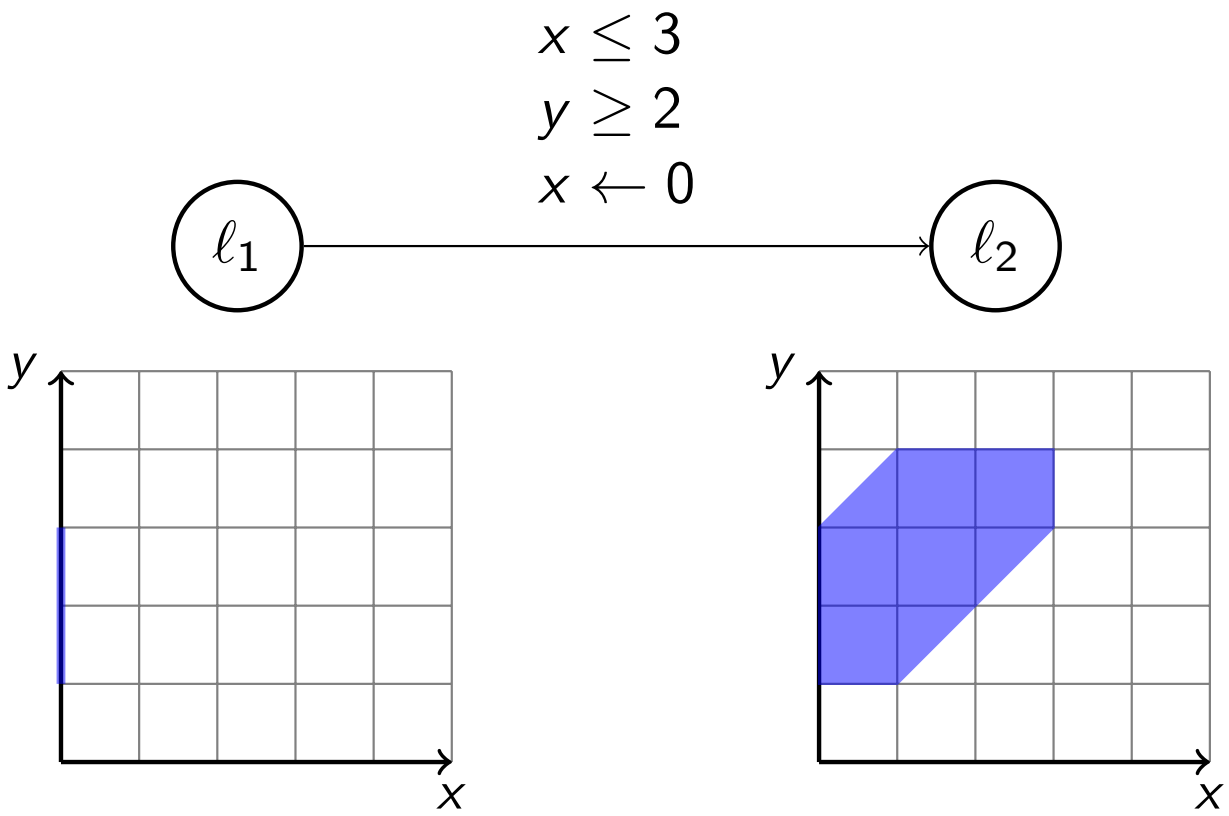

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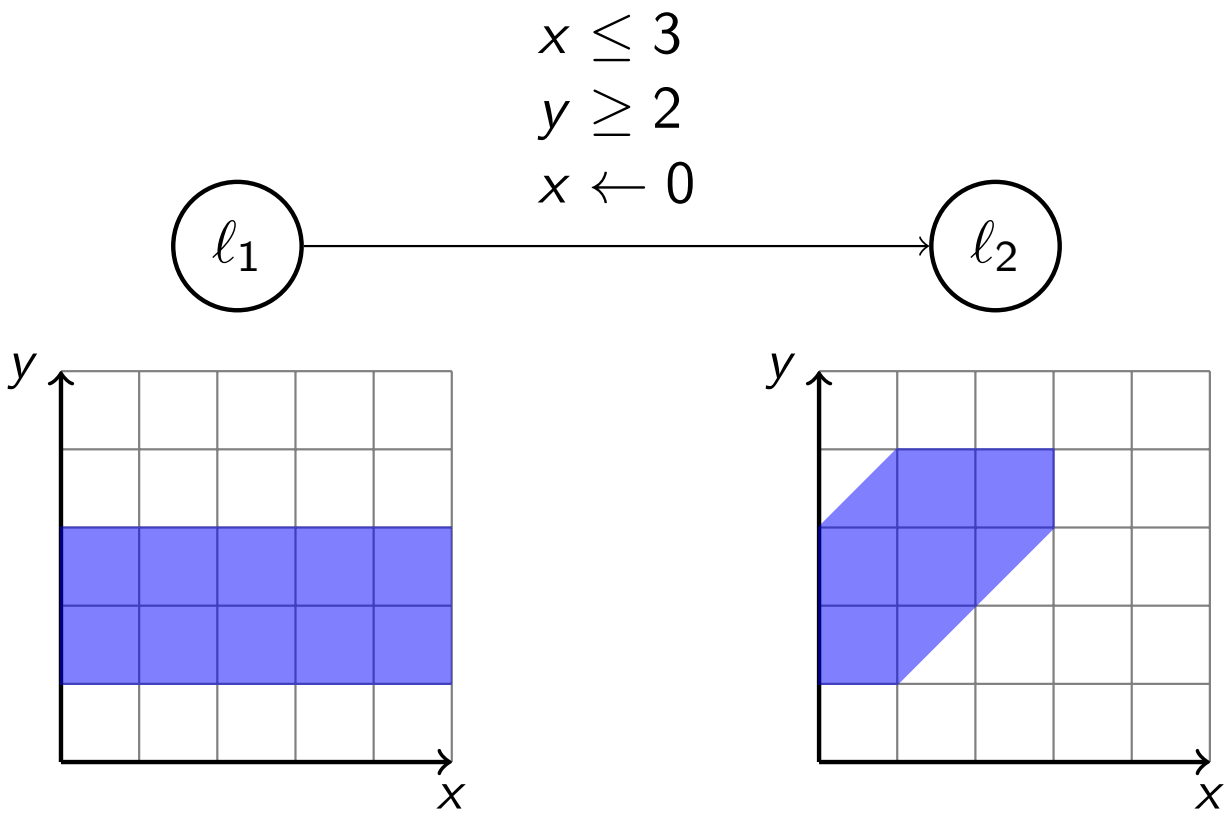

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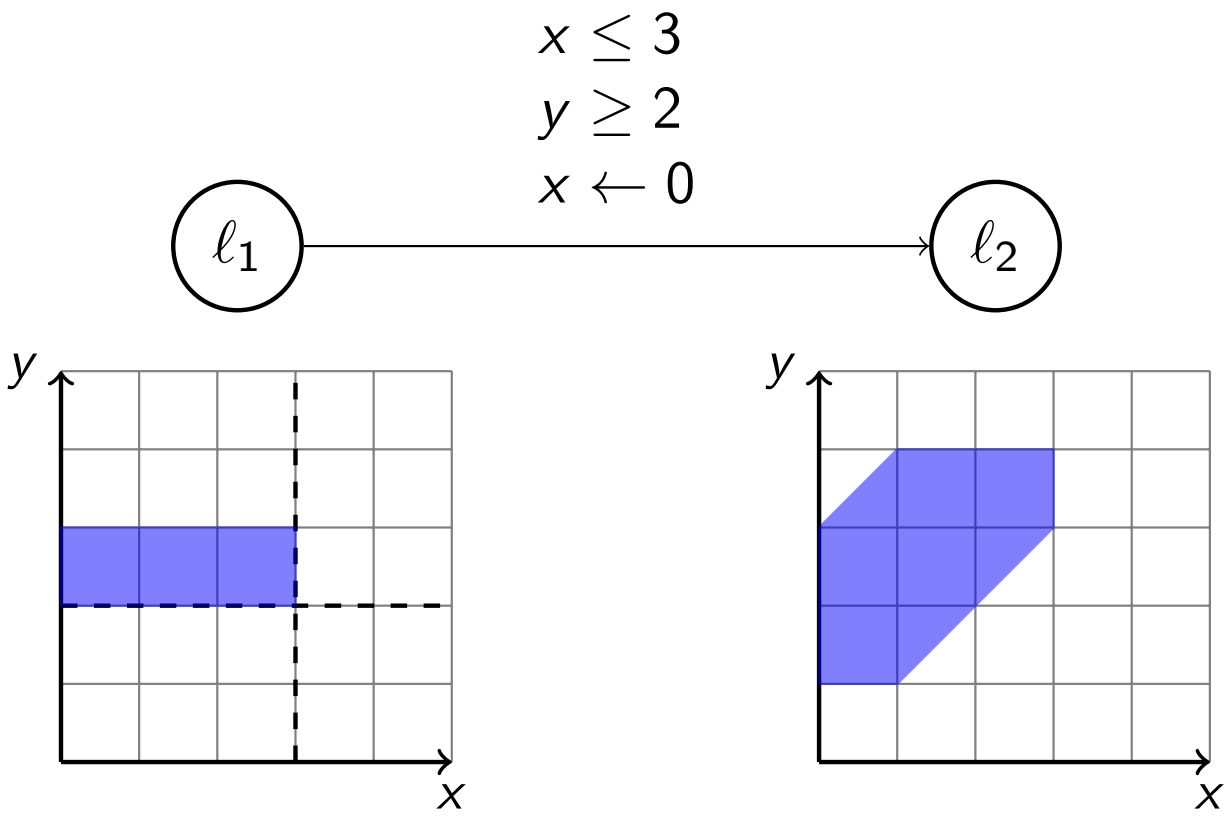

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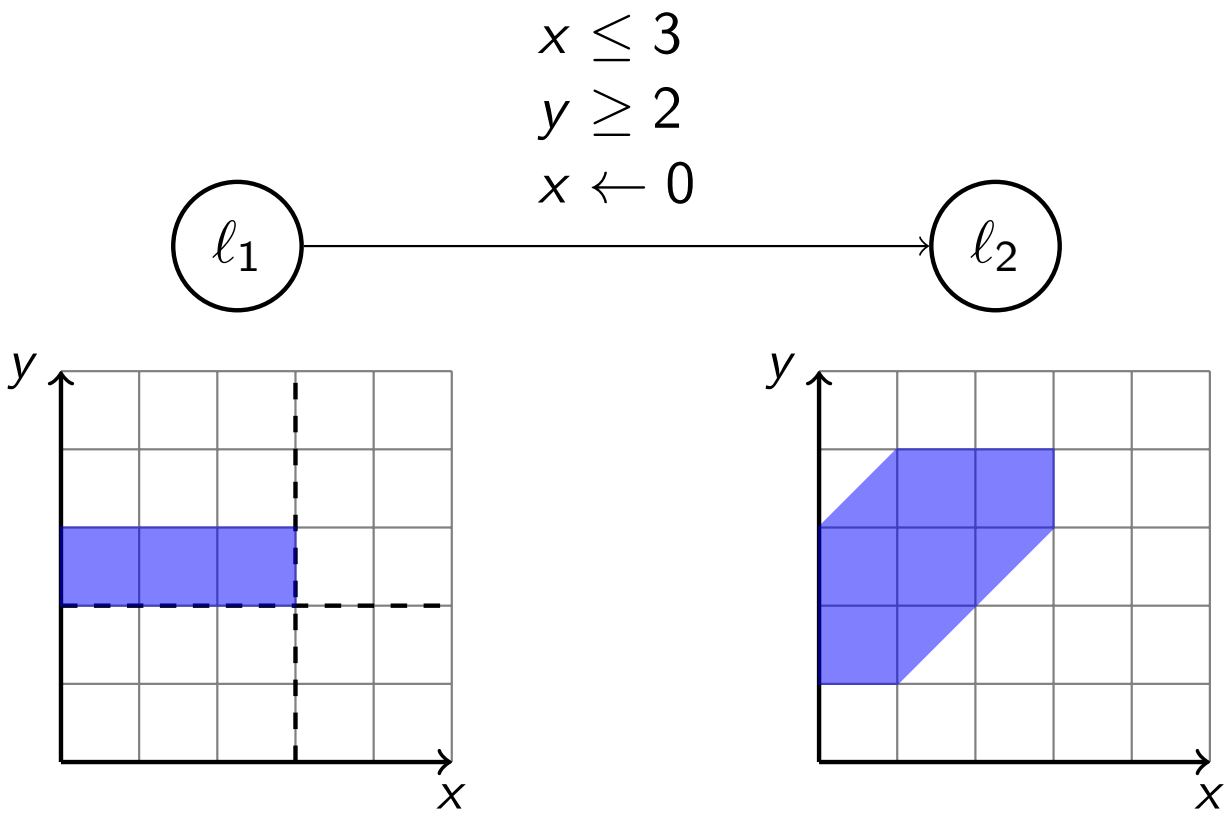
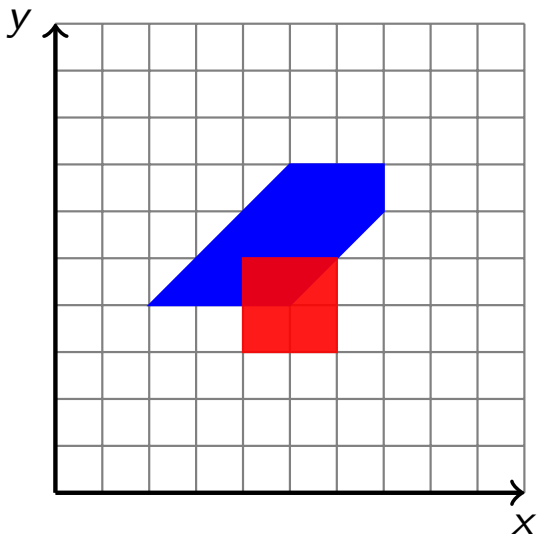


$$Z_1 = \text{unreset}(Z_2 \wedge (x == 0)) \wedge (x \leq 3 \wedge y \geq 2)$$

Computing time predecessors (with zones)



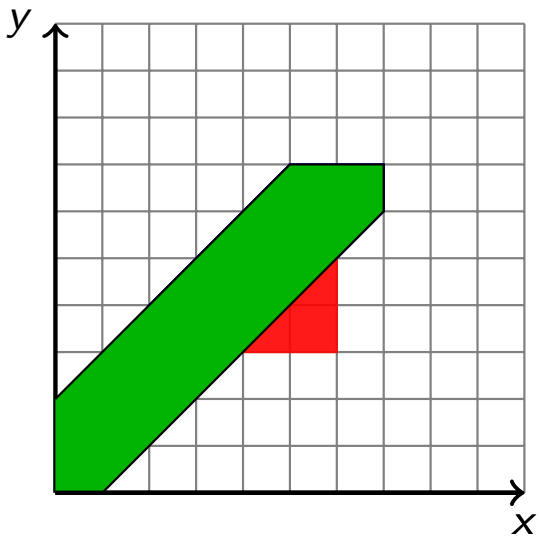
For G convex :

$$\mathrm{Pre_t}(G, B) = (G^{\swarrow} \setminus B^{\swarrow}) \cup (G \cap B^{\swarrow} \cap \overline{B})^{\swarrow}$$

And :

$$\mathrm{Pre_t}(\bigcup_i G_i, \bigcup_j B_j) = \bigcup_i \bigcap_j \mathrm{Pre_t}(G_i, B_j)$$

Computing time predecessors (with zones)



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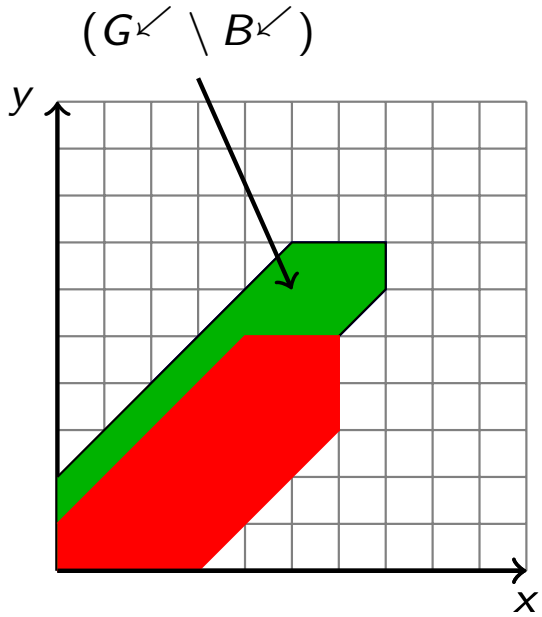

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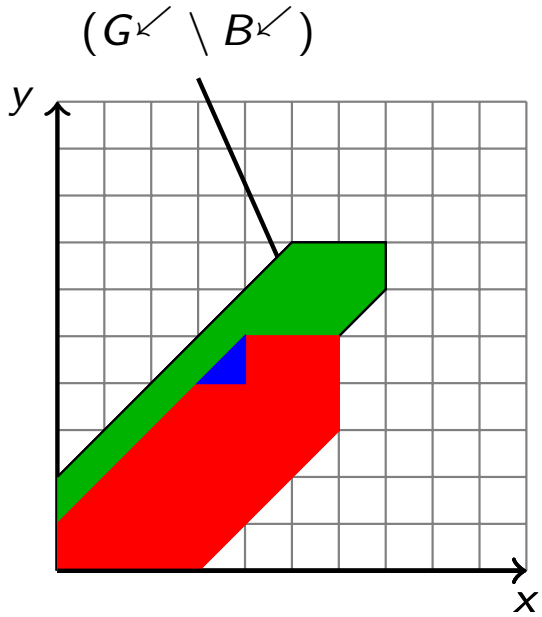

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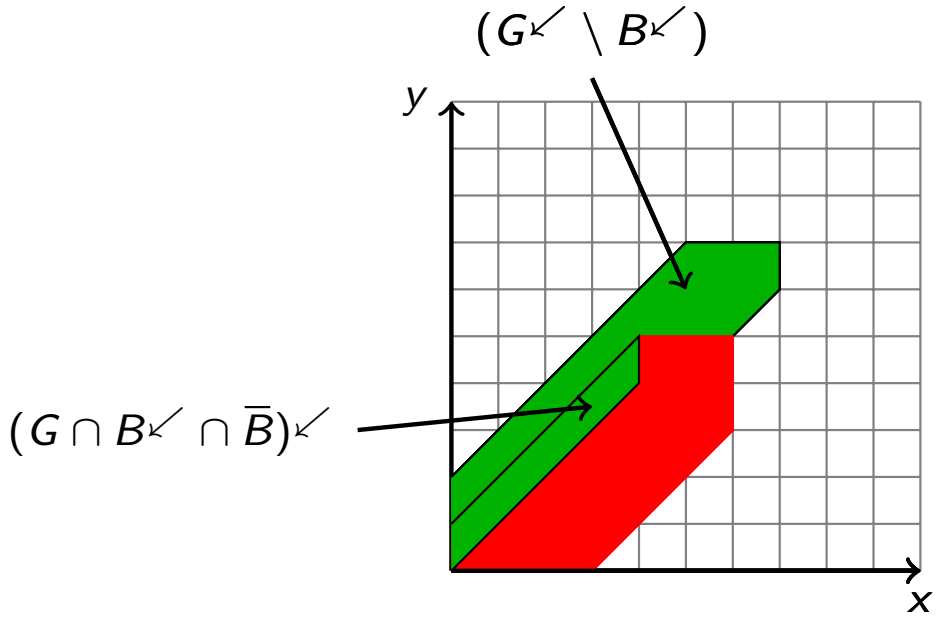

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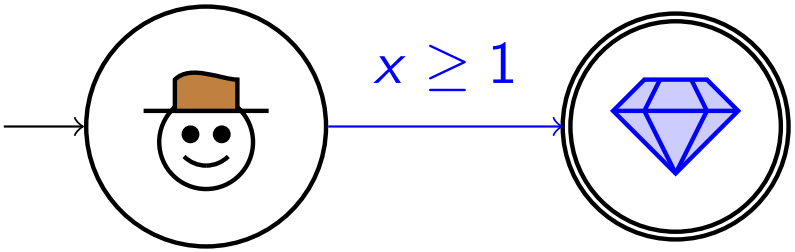
Urgency and invariants

Who wins ?



Urgency and invariants

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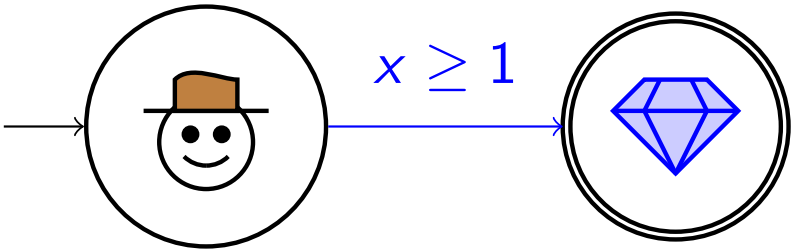

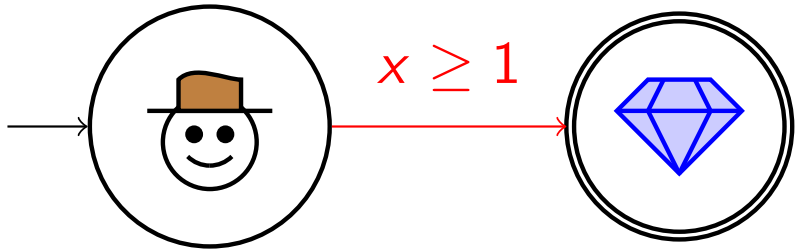

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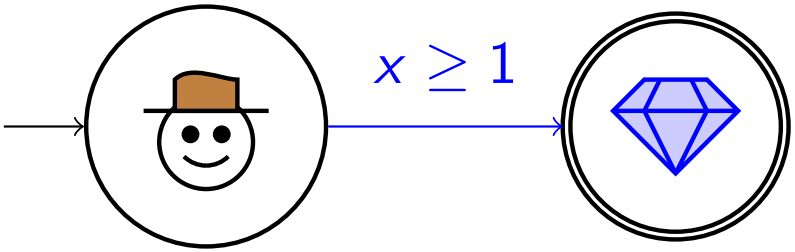
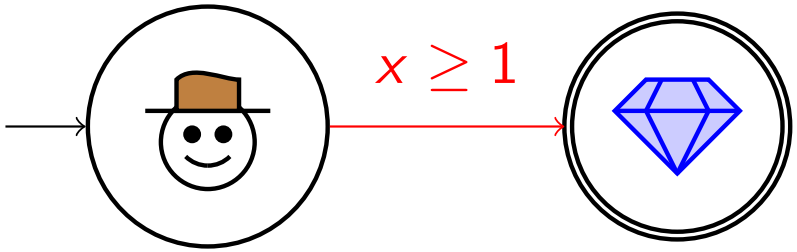
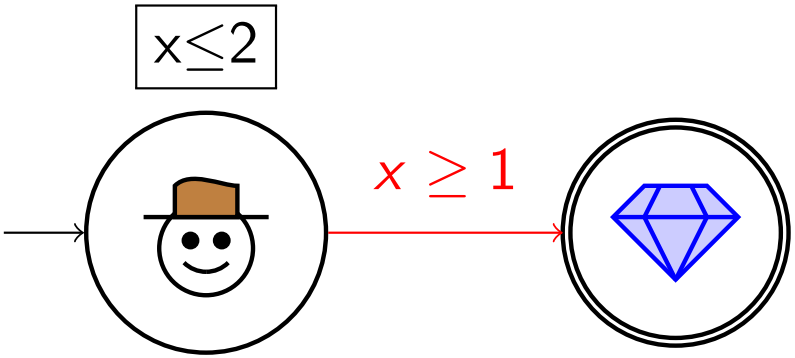
Who wins ?

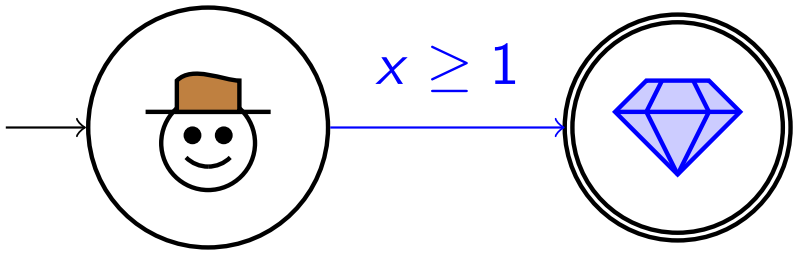


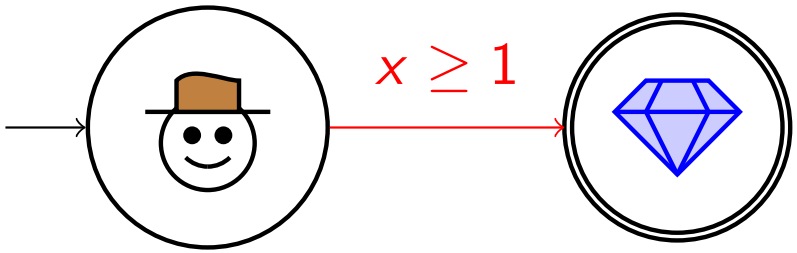


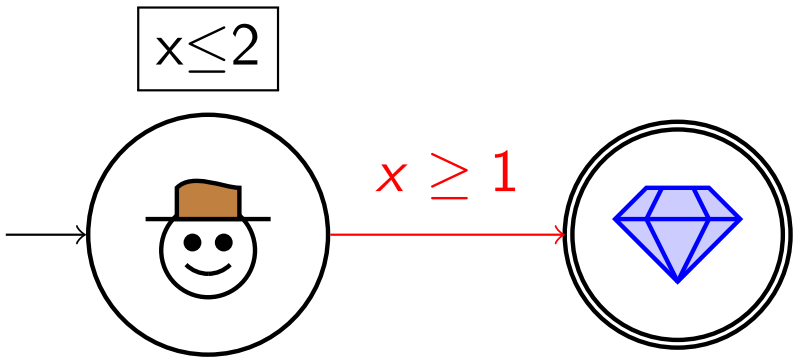


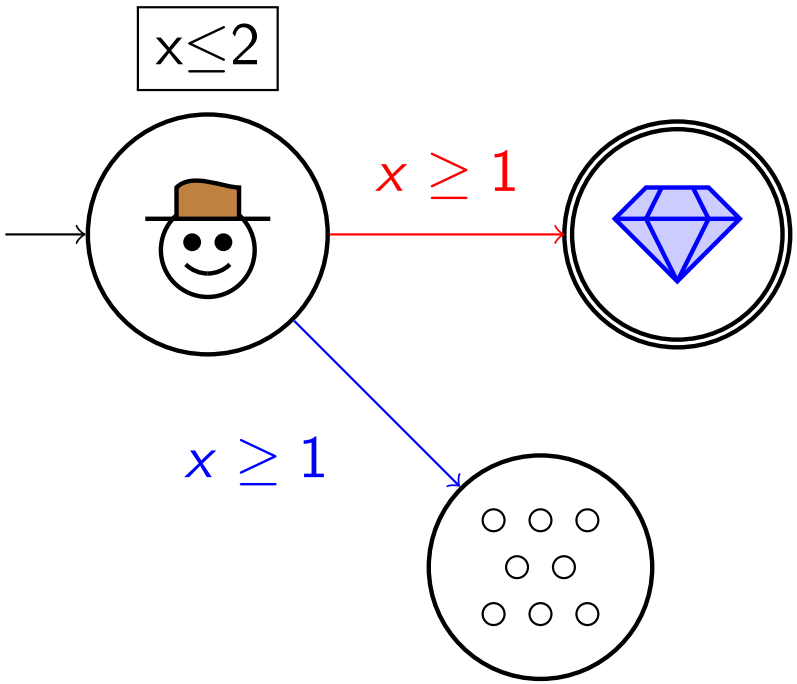
Urgency and invariants

Who wins ?

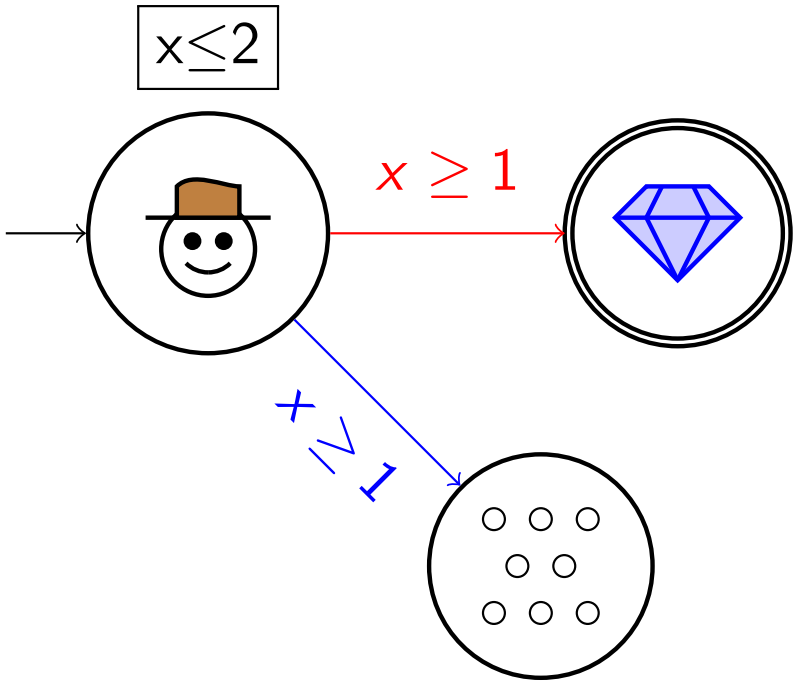




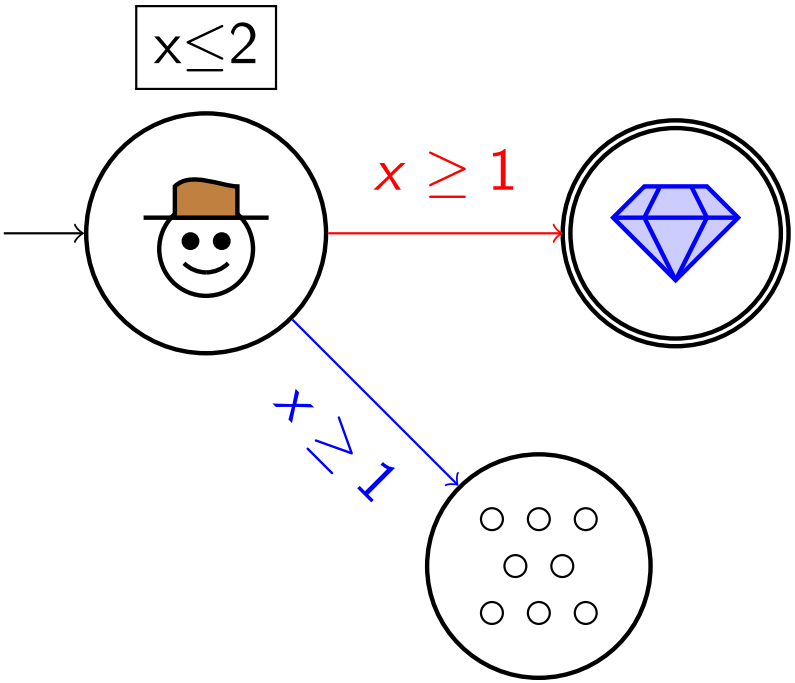


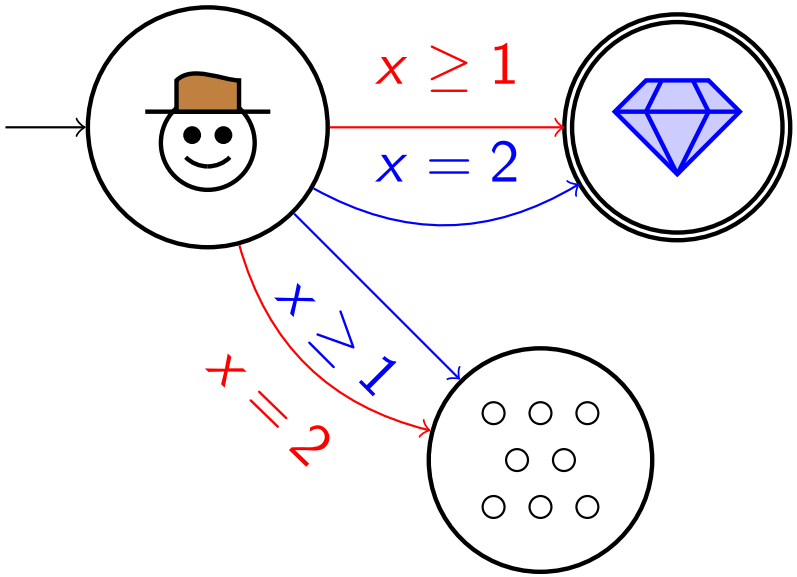


Urgency and invariants

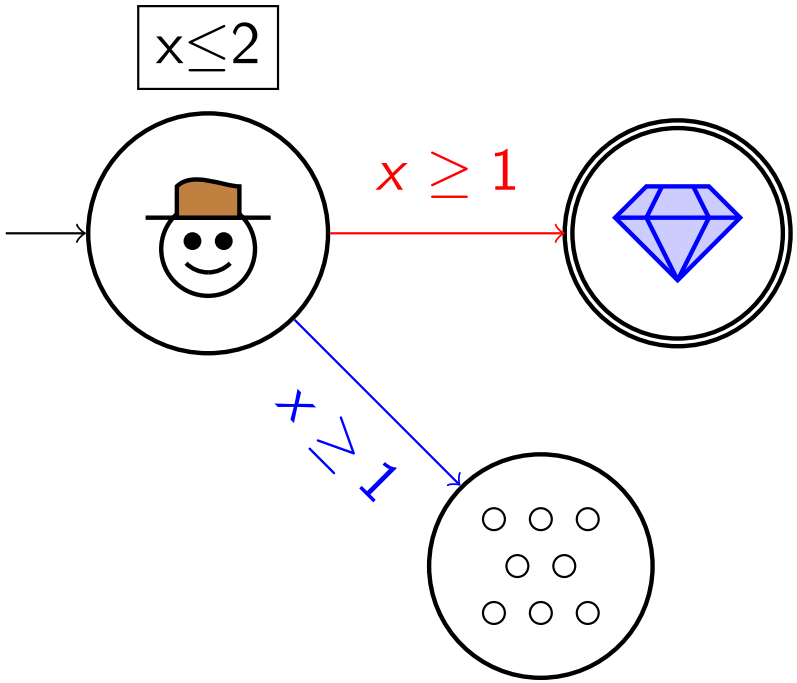


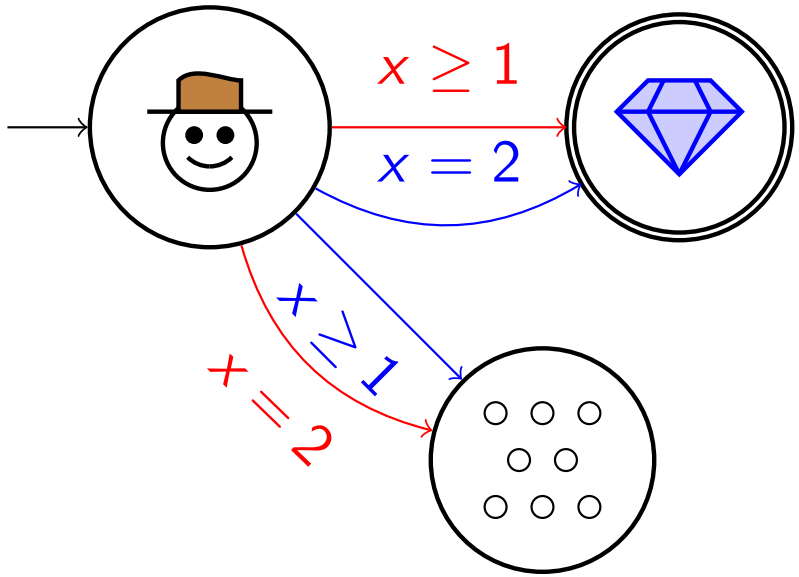
Urgency and invariants





Urgency and invariants





Actually, we compute :

$$\mathrm{Pre_t}\Big(\mathrm{Pre}_c(X) \cup \mathrm{ForcedPre}_u(X), \mathrm{Pre}_u(\overline{X}) \cup (\mathrm{ForcedPre}_c(\overline{X}) \setminus \mathrm{Pre}_c(X))\Big)$$

Safety timed game

- As before, we can solve safety timed games in two different ways
  There are actually subtle differences between the two wrt. when waiting is winning

Safety timed game

- As before, we can solve safety timed games in two different ways
  There are actually subtle differences between the two wrt. when waiting is winning
- Either, we can compute a greatest fixpoint as before :

$$X_0 = G$$
$$X_{n+1} = X_n \cap \pi(X_n)$$

With :

$$\pi(X) = \mathrm{Pre}_t(\mathrm{Pre}_c(X), \mathrm{Pre}_u(\overline{X}))$$

Safety timed game

- Or, we can compute the losing states :

$$Y_0 = \text{Bad}$$
$$Y_{n+1} = Y_n \cup \pi_u(Y_n)$$

With :

$$\pi_u(Y) = \text{Pre}_t^u(\text{Pre}_{u(Y)}, \text{Pre}_c(\overline{Y}))$$

and

$$\text{Pre}_t^u(X, Y) = \{s \in S | \exists d \geq 0, s' \in X \text{ s. t. } s \xrightarrow{d} s' \text{ and } \forall 0 \leq d' < d, s \xrightarrow{d'} \notin Y\}$$

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- Strategy : compute the winning states $W = \overline{Y}$ and,
  - in $\text{Pre}_c(W)$ do the corresponding action ;
  - in the other winning states, wait.

Conclusion

A word about complexity

- Existence of a winning strategy in reachability/safety games :

	one-player	two-player
discrete	NL-complete	P-complete
timed	PSPACE-complete	EXPTIME-complete

Conclusion

- Summary
  - A very expressive framework ;
  - “Easy” to extend, e. g., with time ;
  - Can express different control objectives (e.g, Safety and reachability)
  - Tools for timed systems : Uppaal ($\geq$ 5.0) and Roméo.
- To go further :
  - More complex objectives : parity, Büchi, co-Büchi, etc.
  - Partial observation ;
  - More than two players ;
  - Costs and rewards, cost bounds, optimality, Nash equilibria.

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Crédits