# Systèmes hybrides : Réseaux de Petri continus et hybrides

Formation sur les Systèmes à Événements Discrets (SED)

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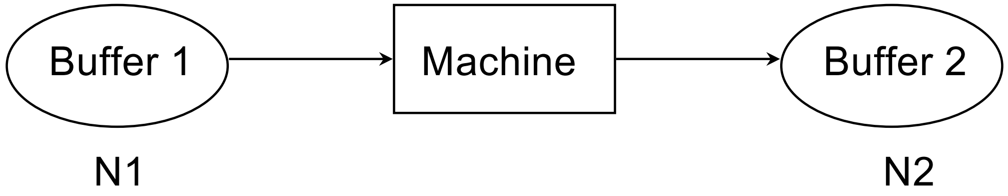
Société d'Automatique,
de Génie Industriel & de Productique

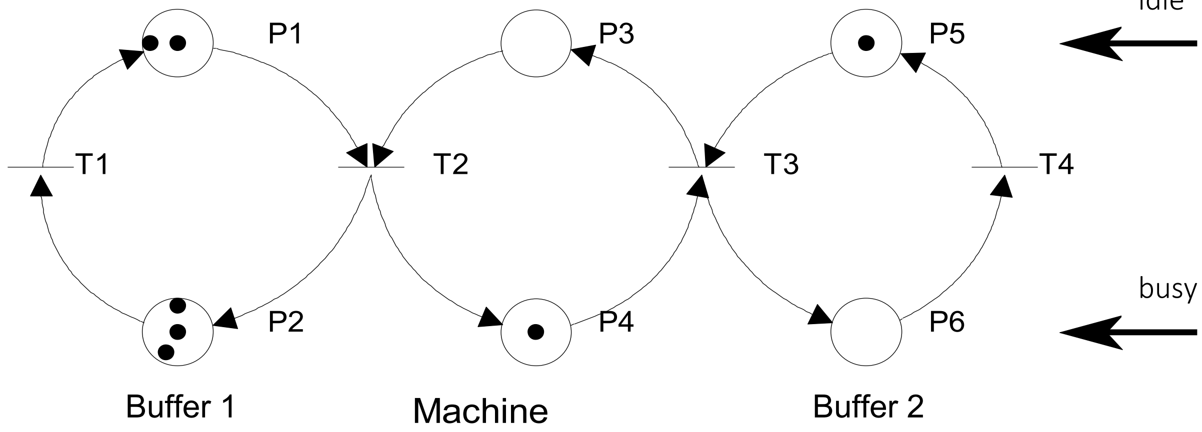
# Introduction

# State explosion : from Discrete PN to Continuous PN

System :




Discrete PN model:



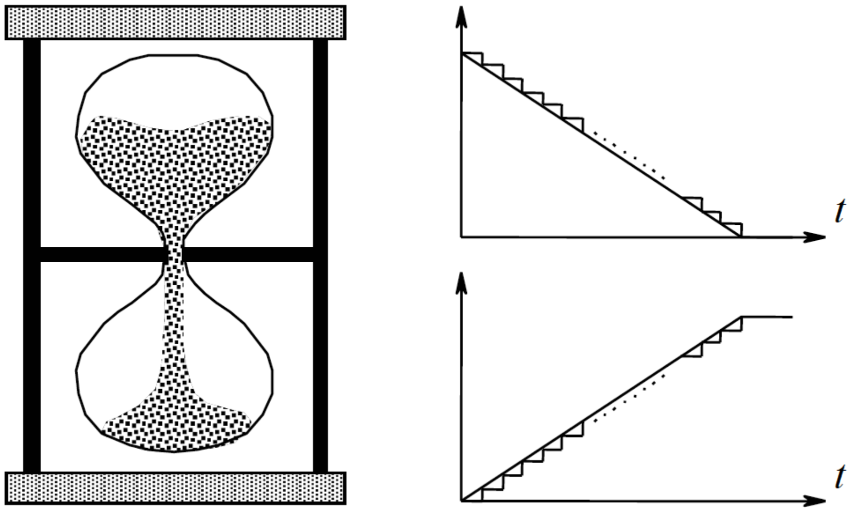

Number of reachable states: N = (N1 +1) (N2 + 1) 2
For N1 = N2 = 10, N = 242. For a production line composed of 10 buffers and 9 machines, $N>10^{13}$ states.

***Motivations***: define a model that is a good approximation of the discrete model and for which the number of reachable states is more reduced.

⇨ Study of marking flows

# Motivation

Large numbers: continuous approximation may be convenient.

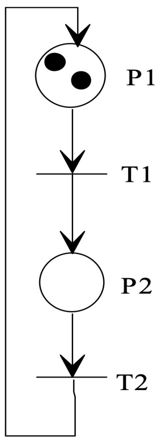


# From discrete PN to autonomous continuous PN

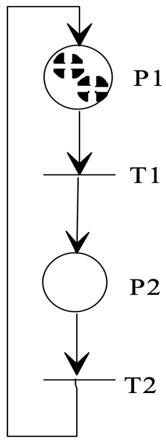
Autonomous continuous PNs are introduced as a limiting case of autonomous discrete PNs.

**Transformation:** dividing each mark into *k* equal parts (tokens)

R = <*Q*, *M*>




Each mark is split into *k* tokens (*k* = 4) →




R'$_{(k)}$= <*Q*, *M'*(*k*)>

$$\begin{bmatrix}2k\\0\end{bmatrix} \underset{T2}{\overset{T1}{\rightleftarrows}} \begin{bmatrix}2k-1\\1\end{bmatrix} \cdots\cdots \begin{bmatrix}k\\k\end{bmatrix} \cdots\cdots \begin{bmatrix}1\\2k-1\end{bmatrix} \underset{T2}{\overset{T1}{\rightleftarrows}} \begin{bmatrix}0\\2k\end{bmatrix}$$

Expressed in tokens

$$\begin{bmatrix}2\\0\end{bmatrix} \underset{T2}{\overset{T1}{\rightleftarrows}} \begin{bmatrix}2-\frac{1}{k}\\\frac{1}{k}\end{bmatrix} \cdots\cdots \begin{bmatrix}1\\1\end{bmatrix} \cdots\cdots \begin{bmatrix}\frac{1}{k}\\2-\frac{1}{k}\end{bmatrix} \underset{T2}{\overset{T1}{\rightleftarrows}} \begin{bmatrix}0\\2\end{bmatrix}$$

Expressed in marks

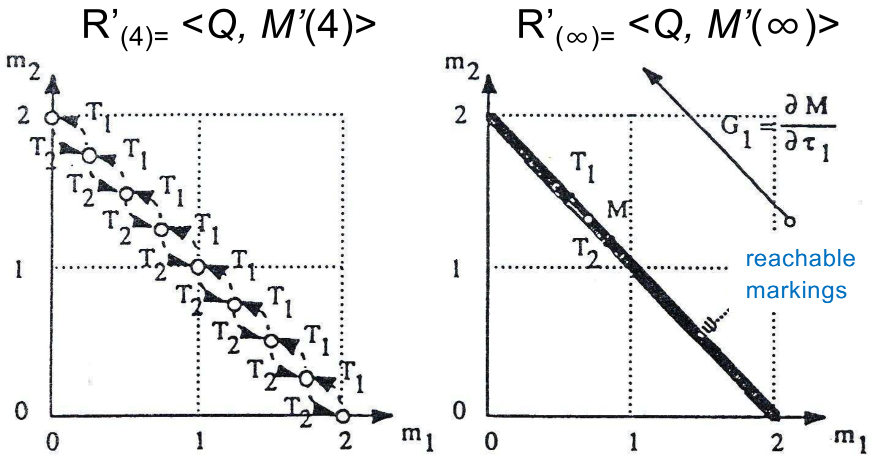
$$\begin{bmatrix}2\\0\end{bmatrix} \underset{T2}{\overset{T1}{\rightleftarrows}} \begin{bmatrix}1\\1\end{bmatrix} \underset{T2}{\overset{T1}{\rightleftarrows}} \begin{bmatrix}0\\2\end{bmatrix}$$

- Expressed in tokens ($m'_i$), the marking is an interger.
- Expressed in marks ($m_i$), the marking is a rational number.

$m_i = m'_i / k$

# Autonomous continuous PN

When $k$ tends to infinity, we obtain an **autonomous continuous Petri net**.




The marking gradient $G_1$ represents the marking variation of the autonomous continuous PN when transition $T_1$ is fired.

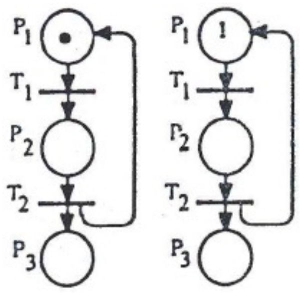
$\tau_1$ is called firing quantity.

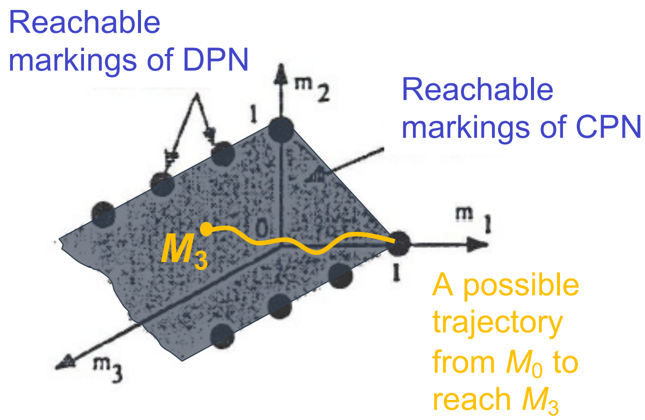
The gradients of marking M with respect to $T_1$ and $T_2$ are the vectors: $G_1 = \frac{\partial M}{\partial \tau_1} = (-1,1)$ and $G_2 = \frac{\partial M}{\partial \tau_2} = (1,-1)$

# Autonomous continuous PN - trajectories

- In a discrete PN, starting from marking $M$, a firing sequence $S$ gives a sequence of successive markings.
- In a continuous PN, starting from marking $M$, a firing sequence gives a trajectory (corresponding to a sequence of successive markings).

*Example.*




For DPN: sequence $S_1 = T_1\ T_2\ T_1$ (i.e., $\underline{S}_1 = (2, 1)$) from $M_0 = (1,0,0)$ reaches $M_1 = (0,1,1)$

For CPN, from $M_0$ to reach $M_1$, possible sequences are:

$S_2 = (T_1)^{0.5}\,(T_1)^{0.5}\,(T_2)^{0.5}\,(T_2)^{0.5}\,(T_1)^{0.5}\,(T_1)^{0.5} = ((T_1)^{0.5})^2\,((T_2)^{0.5})^2\,((T_1)^{0.5})^2$

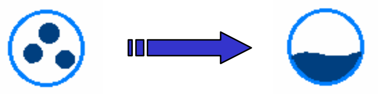
$S_3 = (T_1)^{0.8}\,(T_2)^{0.6}\,(T_1)^{0.5}\,(T_2)^{0.4}\,(T_1)^{0.7}$ with $0.8+0.5+0.7 = 2$ and $0.6+0.4 = 1$

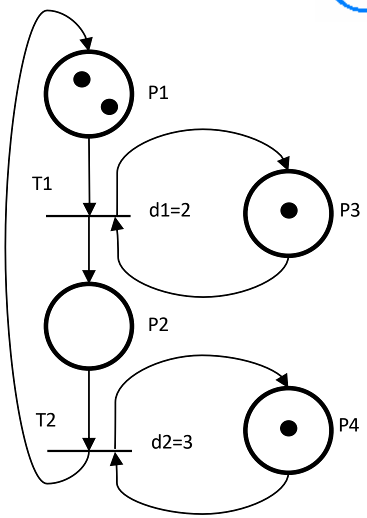
Characteristic vector of a trajectory: vector where each component is a real number corresponding to the firing quantity of the corresponding transition. $\boldsymbol{M' = M + W.\underline{S}}$

*Remark.* There is an infinite number of trajectories which have the same characteristic vector.

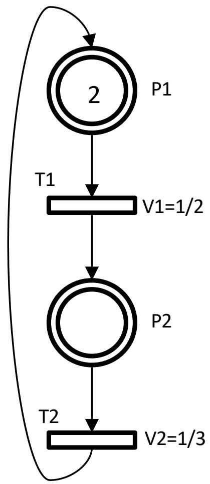
*Liveness: yes or not?: lim-reachability (!)*

# From T-Timed discrete PN to Timed continuous PN






- Timed T discrete PN with *k* serveur semantics

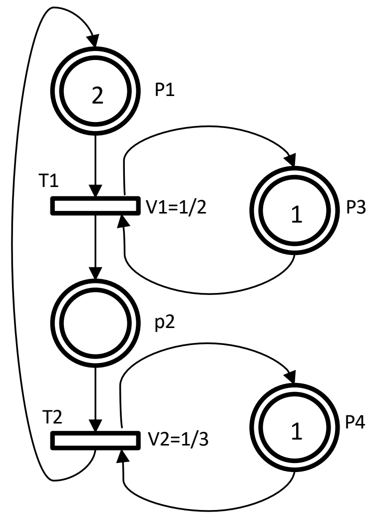



- **Constant continuous PN** (CCPN)

or CPN with finite servers semantics

$v_1 = V_1 = k / d_1$

$v_2 = V_2 = k / d_2$




- Variable continuous PN (VCPN) or CPN with infinite servers semantics

$v_1 = V_1 \min(m_1, m_3)$

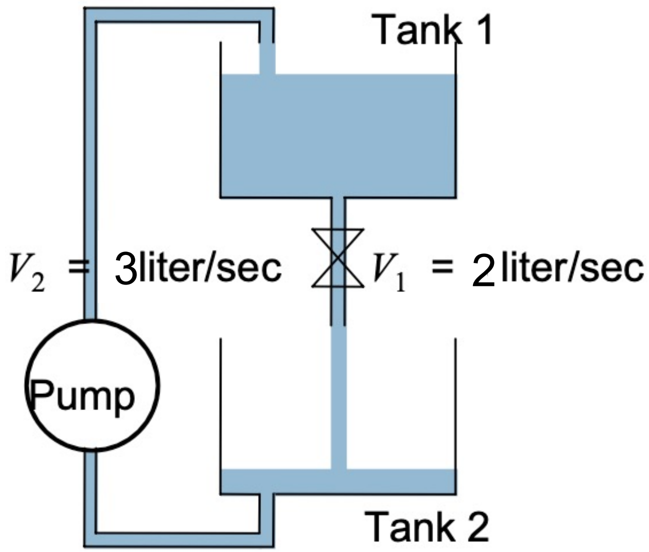
$v_2 = V_2 \min(m_2, m_4)$

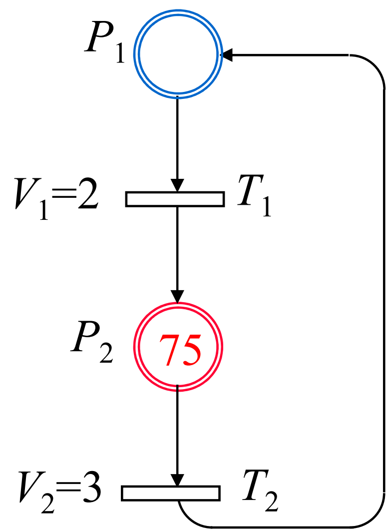
# Constant Continuous Petri nets

# Constant Continuous PN (or CPN with finite server semantics)

- **Continuous place $P_i$**: $m_i$ is a non negative real
- **Continuous transition $T_j$**: $V_j$ is called maximal speed and $v_j$ is called instantaneous firing speed.

*Example.*



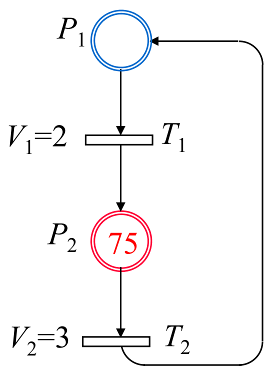



# Enabling of a transition

- A continuous place is fed at a time $t$ if this place has a null marking but has at least one input transition with a non-zero instantaneous firing speed.
  $m_i(t) = 0$ $\quad P_i$ *is fed if :* $\quad \exists\, T_j \in {}^\circ P_i$ and $v_j(t) \neq 0$

- *Enabling conditions for a continuous transition:*
  - A continuous transition is strongly enabled at time $t$ if the marking of all its input places is strictly positive.
    $T_j$ is strongly enabled if $\forall\, P_k \in {}^\circ T_j \,/\, m_k(t) > 0$
  - A continuous transition is weakly enabled at time $t$ if each of its input places that has a null marking is fed.
    $T_j$ is weakly enabled if $\exists\, P_k \in {}^\circ T_j \,/\, m_k(t) = 0$ *and* $P_k$ *is fed.*

*Example.*




For the initial marking, $M_0$ = (0, 75), at time $t_0$:
$T_2$ is strongly enabled => $v_2(t_0) \neq 0$
Thus place $P_1$ is fed.
$T_1$ is then weakly enabled.

# Instantaneous firing speed

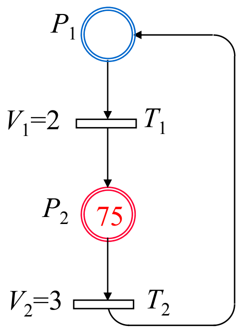
- The dynamic balance $B_i$ of place $P_i$ corresponds to the marking gradient:

$$B_i(t) = \dot{m}_i(t) = \sum_{k=1}^{n} Post(P_i, T_k).v_k(t) - \sum_{k=1}^{n} Pre(P_i, T_k).v_k(t)$$

It represents the increasing or decreasing variation of the marking and cannot be negative.

- *Instantaneous firing speed of a continuous transition:*
  - For a not enabled transition: $v_j(t) = 0$
  - For a strongly enabled transition: $v_j(t) = V_j$
  - For a weakly enabled transition: $v_j(t) = \min\left[V_j\ ,\ \min_{P_i \in {}^{\circ}T_j / m_i(t)=0} \left( B_i(t) + Pre(P_i, T_j).\, v_j(t)\right)\right]$

*Example.*




$T_2$ is strongly enabled => $v_2(t_0) = V_2 = 3$

$P_1$ is fed => $B_1(t_0) = v_2(t_0) - v_1(t_0) = 3 - v_1(t_0)$

$T_1$ is weakly enabled => $v_1(t_0) = \min(V_1, 3 - v_1(t_0) + v_1(t_0)) = \min(2, 3) = 2$.

Consequently, $B_1(t_0) = 1$ and $B_2(t_0) = -1$.

# State equation

The firing of transition $T_j$ at time $(t+dt)$ leads to the new marking such that:

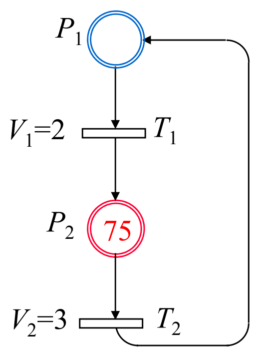
$\exists\, P_i \in {}^{\circ}T_j$ $\quad m_i(t+dt) = m_i(t) - v_j(t)\,.\,Pre(P_i, T_j).\,dt$

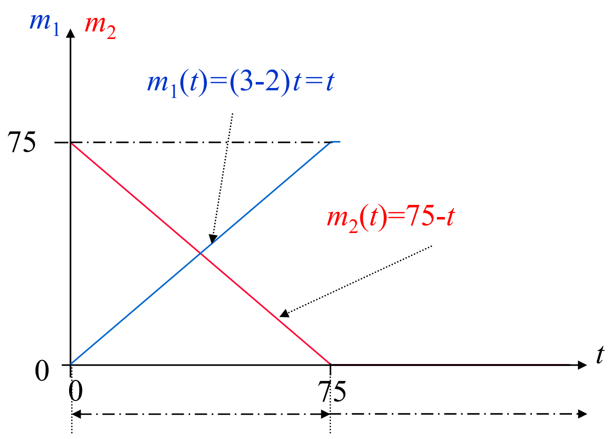
$\exists\, P_i \in T_j^{\circ}$ $\quad m_i(t+dt) = m_i(t) + v_j(t)\,.\,Post(P_i, T_j).\,dt$

- State equation $M(t) = M(t_O) + W\,.\int_{t_0}^{t} v(u)du$ or $\left.\dfrac{dM(u)}{du}\right|_t = W\,.v(t) = B(t)$

⚠ Implicite equation => use fixed point iterative method or linear programming

*Example.*



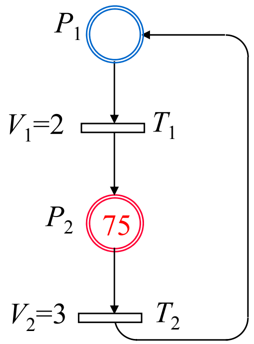



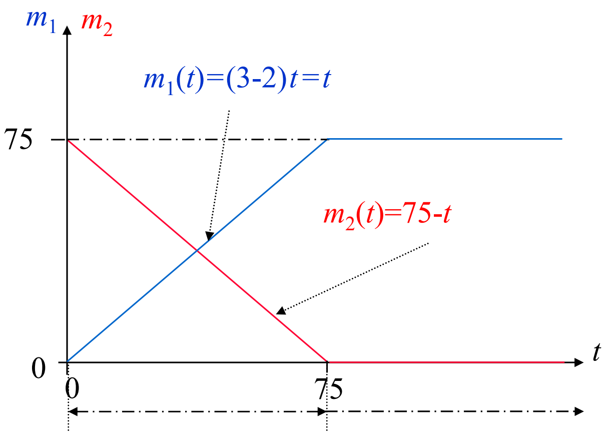
# Evolution graph

- An invariant-state behaviour (IB-state ) is defined by the instantaneous firing speed vector that remains constant during two timed events.

- An event occurs when a marking of a place (with a negative balance) becomes equal to zero.

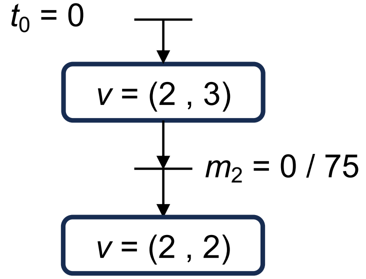
*Example.*




$M_0$ = (0, 75)

$M_1$ = (75, 0)




$B_0$ = (1, -1)

$B_1$ = (0, 0)

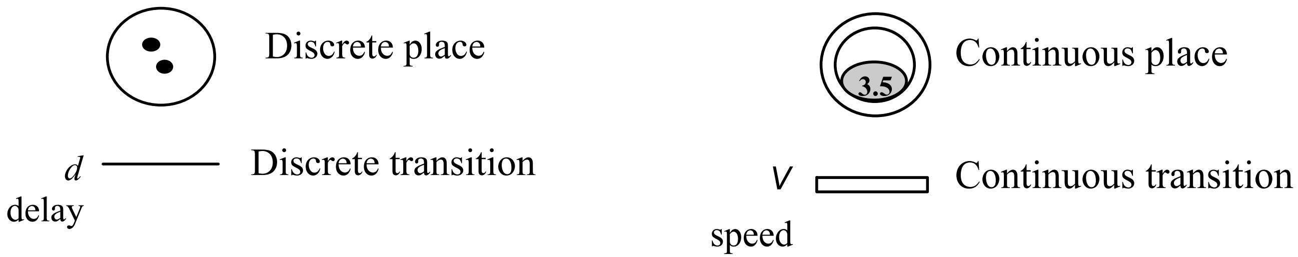
# Hybrid Petri nets

# Hybrid Petri nets

(Timed) Hybrid Petri net = T-timed discrete PN U Constant continuous PN

An hybrid Petri net is defined by $H = < R, f, M_0 , Tempo >$ where:

1) $R$ is a Petri net defined by: $R = < P, T, Pre, Post >$
2) $f : P \cup T \rightarrow \{ D, C\}$ called "hybrid function", indicates whether each node is discrete or continuous.
3) $M_0$ is the initial marking.
4) *Tempo* is the application that associates a non-negative rational number with each transition.



# Fundamental equation

## Marking

$$M(t) = \begin{pmatrix} M^D(t) \\ M^C(t) \end{pmatrix} = \begin{pmatrix} \begin{bmatrix} m_1^D(t) \\ \vdots \\ m_{n_d}^D(t) \end{bmatrix} \\ \begin{bmatrix} m_1^C(t) \\ \vdots \\ m_{n_c}^C(t) \end{bmatrix} \end{pmatrix} \in \begin{pmatrix} [\mathbb{N}^{n_d}] \\ [\mathbb{R}^{+n_c}] \end{pmatrix}$$

*Remark*: reserved policy, *i.e.*, the marking is decomposed into a reserved marking $M^r$ and a non reserved marking $M^n$

$$m_i(t) = m^r_i(t) + m^n_i(t)$$

## Incidence matrix

$$W = \begin{bmatrix} W^{DD} & 0 \\ W^{CD} & W^{CC} \end{bmatrix} \begin{matrix} P^D \\ P^C \end{matrix}$$

$$\qquad T^D \qquad\qquad T^C$$

*Remark*: if $f(P_i) = D$ and $f(T_j) = C$ then $Pre(P_i, T_j) = Post(P_i, T_j)$. This structural condition ensures that the marking of discrete places remains an integer, whatever the evolution that occurs.

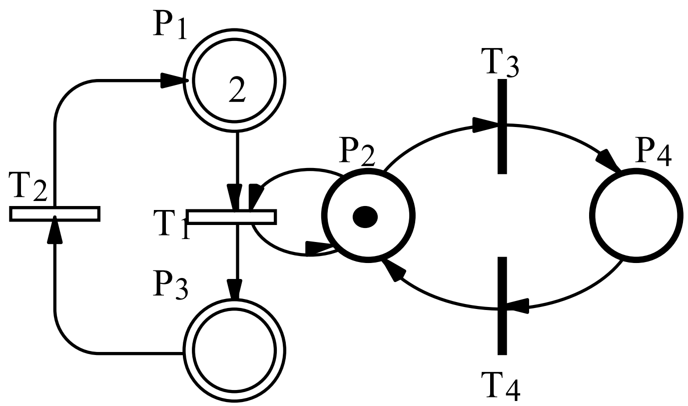
## Fundamental equation

$$M(t) = M(t_O) + W \,.\, (\sigma(t) + \int_{t_0}^{t} v(u)du)$$

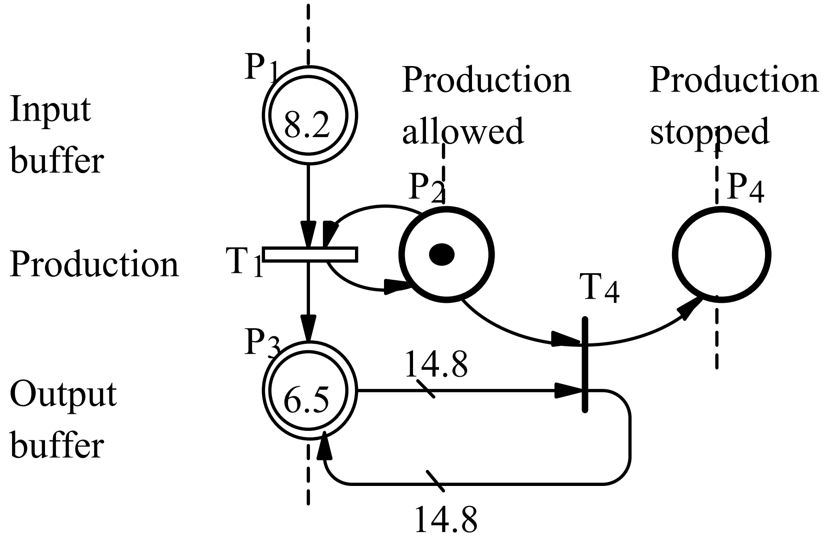
*with* $\sigma(t)$ : firing vector
$v(t)$ : instantaneous firing speed vector

# Influences Discrete - Continuous

Influence of the discrete part on continuous part

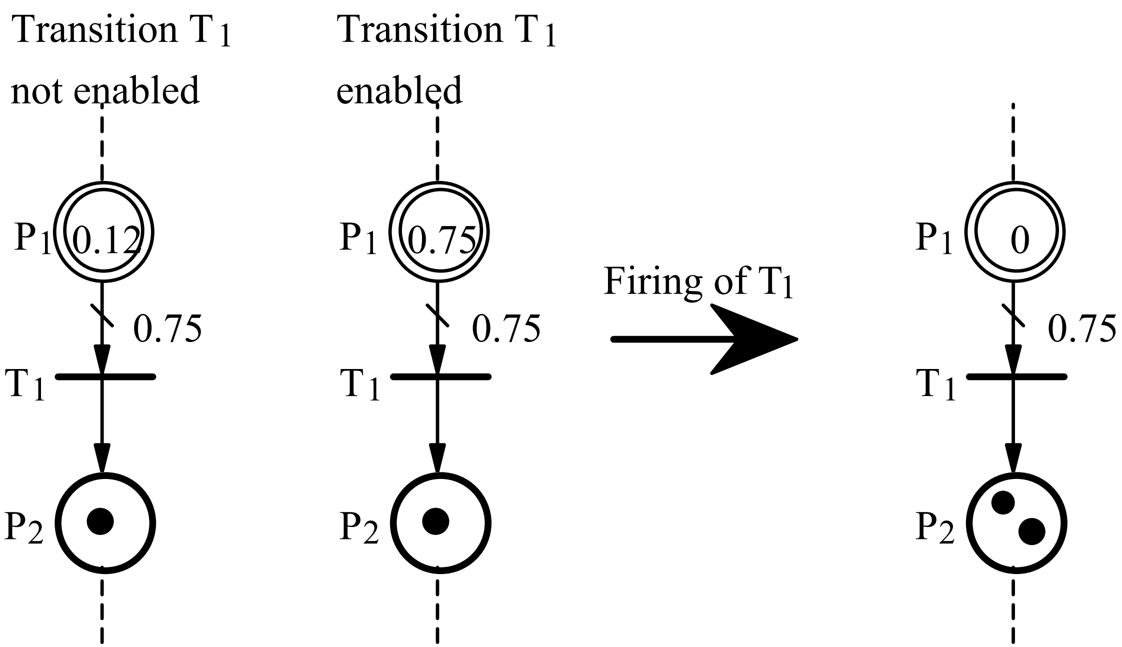



Influence of the continuous part on the discrete part

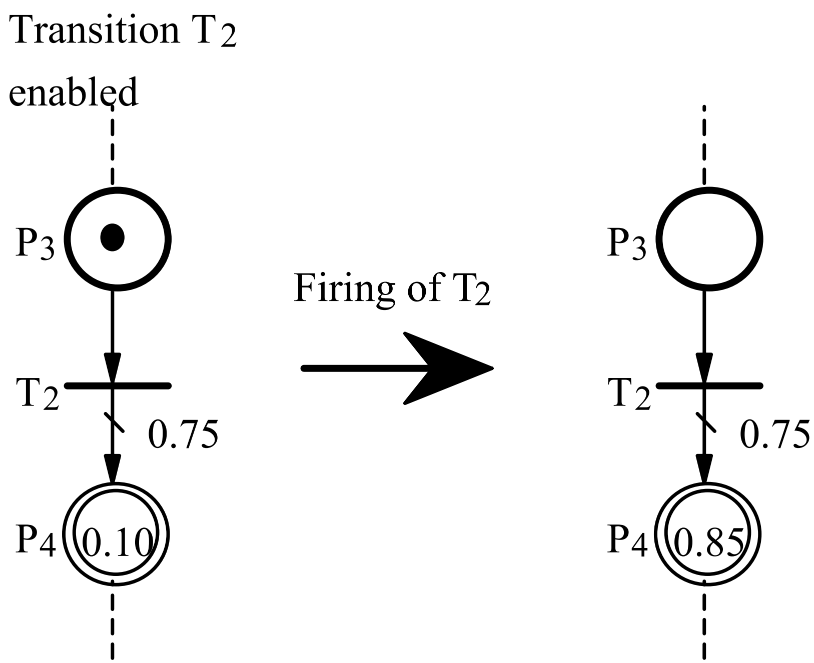


# Transformations Discrete - Continuous

Transformation continuous marking into discrete one

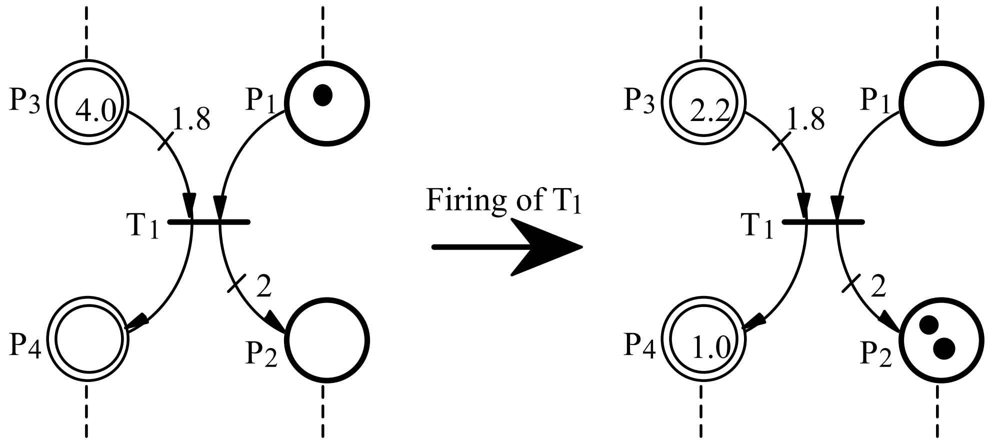



Transformation discrete marking into continuous one

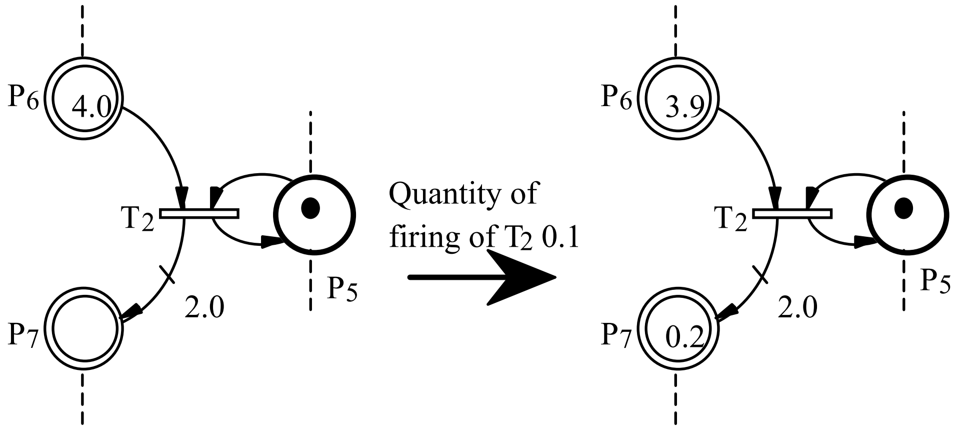


# General case

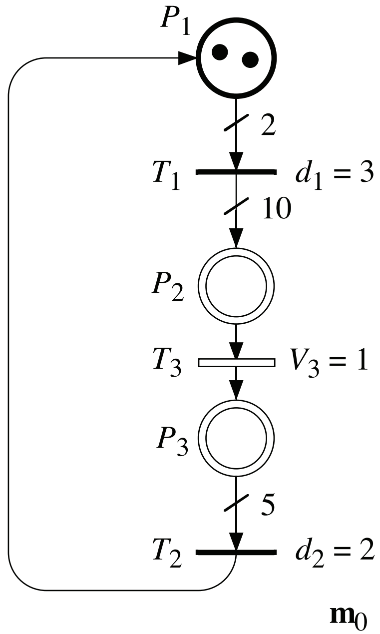
Firing of a discrete transition

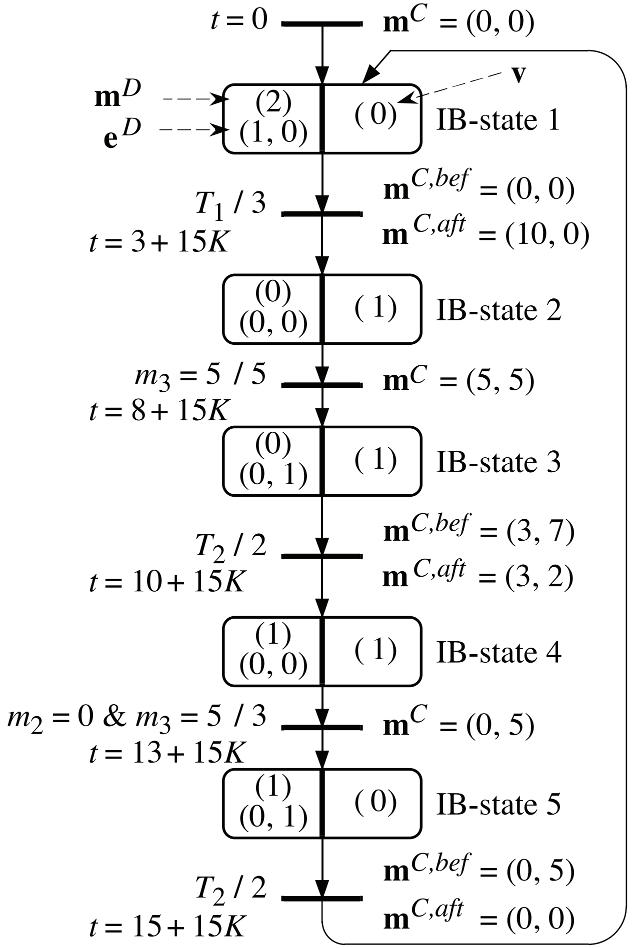



Firing of a continuous transition



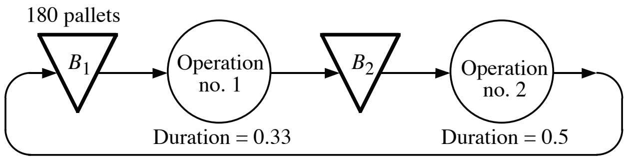
# Evolution graph

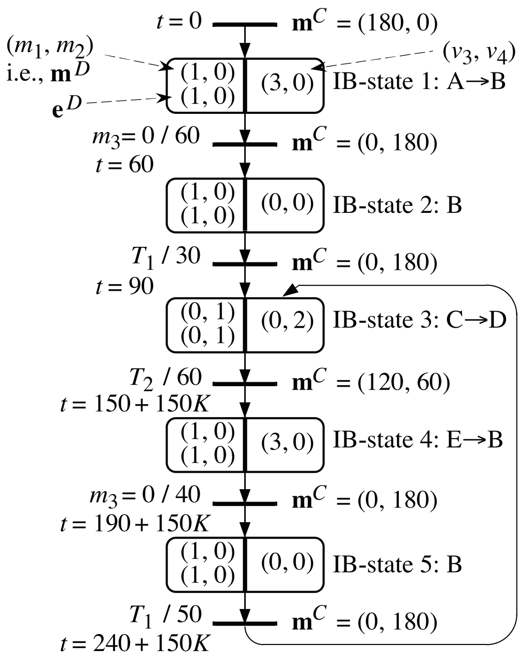


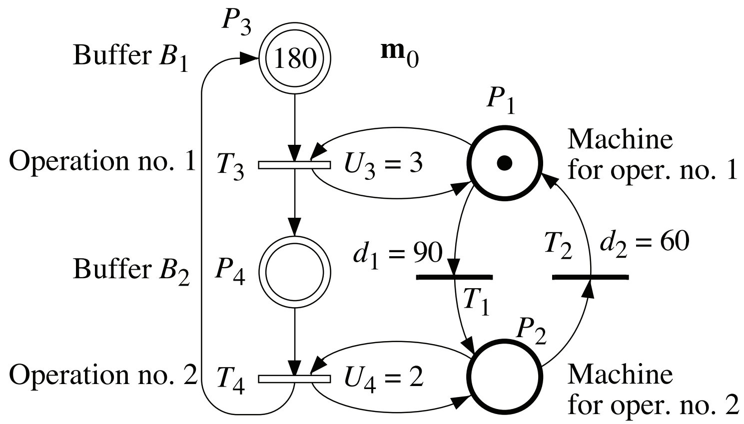
# Example of hybrid PN

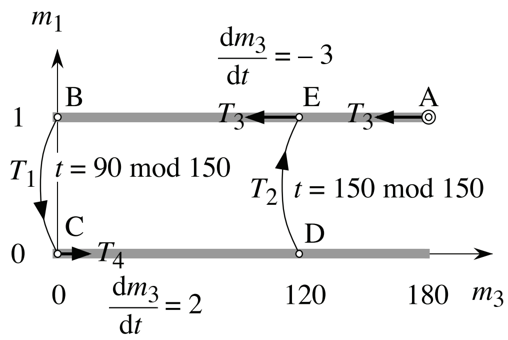
Production system with one machine for both operations:




Hybrid PN model



# Conclusions

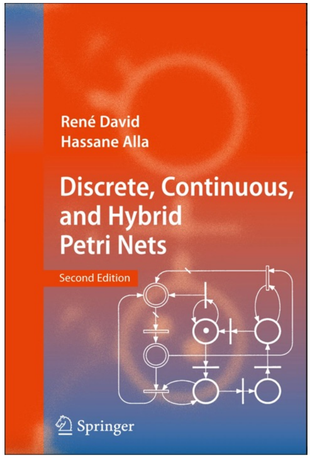
# Conclusions

Several type of Petri nets:

- **autonomous**
- labeled
- synchronized
- **timed**, **time**
- stochastics
- colored, object, fuzzy
- **continuous, hybrid**, batch
- etc.

# Références bibliographiques

# Peers and fathers in continuous and hybrid PNs




David, R. and Alla, H. (2010).
*Discrete, Continuous, and Hybrid Petri Nets.*
Springer.

René David
(1939-2022)

Hassan Alla
(1952 - )

Manuel Silva
(1951-2022)

# Remerciements et crédits

# Équipe pédagogique

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# Crédits